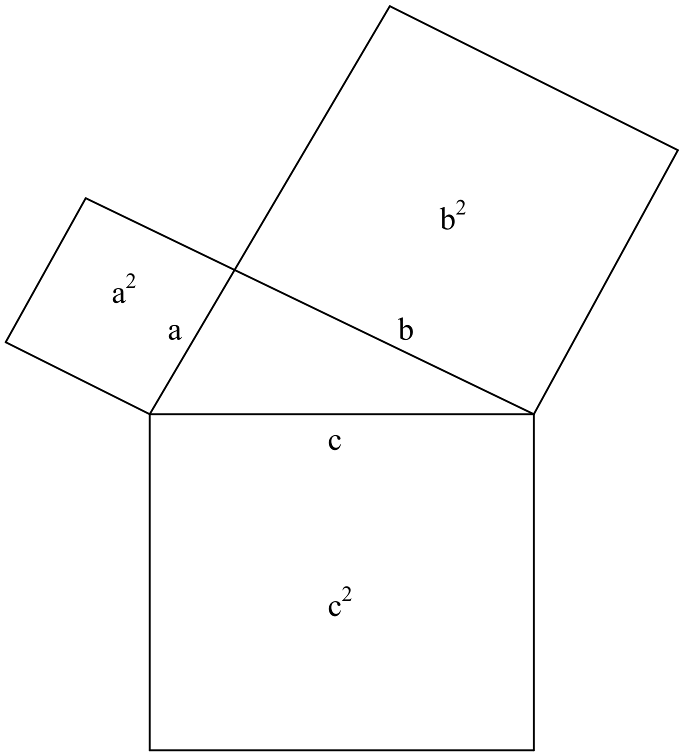
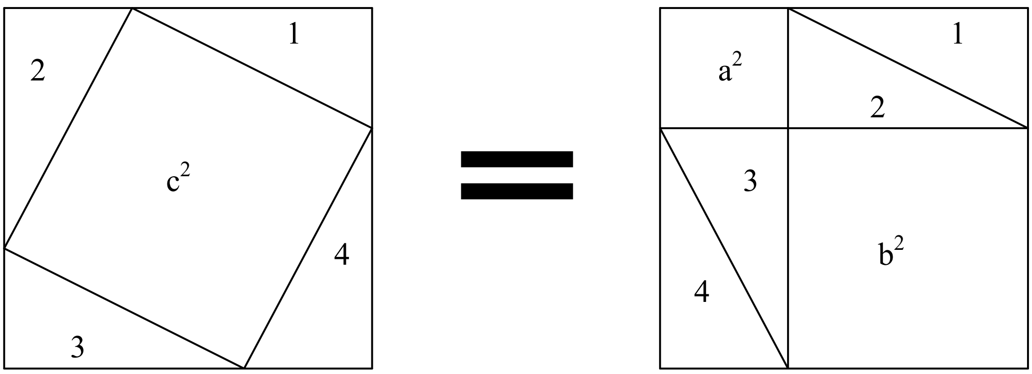
Behind The Pythagoras Theorem

The theorem claims that if we draw squares on the sides of a right angle triangle, then the two smaller ones together equal the biggest:



The Hindu proof applies the trick of adding four copies of the triangle first to c^2 and then to $a^2 + b^2$ and obtaining in both cases an $a + b$ sided big square:



Thus of course, both $a^2 + b^2$ and c^2 are $(a + b)^2$ minus the four triangles.

Which algebraically can also be calculated as $(a + b)^2 - 4\frac{ab}{2} = (a + b)^2 - 2ab = a^2 + b^2$.

So strangely, the second geometrical construction is verified by it.
This little post script of course was not part of the Hindu proof.
Indeed, the whole point of that approach was using only intuitively obvious visual steps that were accepted by the listener's noddings.
This is in sharp contrast to the Euclidean or axiomatic process.
Here if we strictly list all the accepted assumptions or axioms, then the proof would be a chain of logical steps, using only the axioms or already proved facts.
Writing this chain down as a derivation, we wouldn't even need pictures.

I leave now this grand duality and venture into a second duality behind the theorem that still won't be our main subject. This one is connectable to Pythagoras himself.
Though this most basic geometrical theorem bears his name, his obsession was the natural numbers. Yet, he did not recognize any true number theoretical facts either.
He was quite simply a numerologist mystic.
His cult worshipped the number ten yet didn't realize the base ten number system.

But back to the second duality behind the Pythagoras Theorem, we can start historically.
And Pythagoras' personal life fits into this perfectly.
He went to Egypt and spent a long time there. Befriended a pharaoh and he was the only Greek allowed in their secret rituals. But the Persians conquered Egypt and he was taken to Babylon.
Strangely, he found friends again and probably learned about the theorem there.
The Babylonians knew it for hundreds of years already.
So very probably they discovered it independently of the Hindus and their approach was very different too. Whether the above Hindu proof was known to Pythagoras is not even certain.
The Babylonian approach was more akin to Pythagoras' number mysticism.

The simplest right angle triangle could be describes as: (1 , 1 , c)
Meaning that we take the length a as 1 and use the same as b and ponder what c is.
Then since this is not evident at once, we instead regard something else like:
(1 , 2 , c) or (1 , 3 , c) or (2 , 3 , c) or (3 , 4 , c) and so on.

And this last case is actually the first "bingo" situation!
Simply because here c must be 5. So the full triplet is (3 , 4 , 5)
And this had to be how the theorem was first recognized. Namely:
The ancient Babylonians drew two perpendicular lines and from the crossing measured on one line 3 distance while on the other 4 and realized that the connection of the ends magically became exactly 5. They obviously knew that a can be anything so believed in a proportionally expandable space and thus using 2 as a would give the obvious (6 , 8 , 10) triplet.
But they also realized a second "original" triplet as (5 , 12 , 13).
All these have an obvious practical side as using distances to make a perfect right angle.
Indeed, we can use two sticks of 3 and 4 length connected at one ends and spread them until a stick of 5 length fits connecting the other ends. And then we get a sure right angle.
Similarly, using 5 and 12 long sticks we must open them till a 13 long stick connect them.
So we have quite flexible lengths to establish perpendicularity.

The two obvious question were why these particular triplets give perpendicular angles and are there infinite many of them. The first was replaced by the mere recognition that:
$3^2 + 4^2 = 9 + 16 = 25 = 5^2$ and then again $5^2 + 12^2 = 25 + 144 = 169 = 13^2$.
The infinity question was then translated into this, that is whether infinite many such original perfect square triplets exist. But they didn't perfectly solve this question either.
An instantly visible fact is that our c value must itself be a sum of squares.
Indeed, $5 = 1 + 4 = 1^2 + 2^2$, $13 = 4 + 9 = 2^2 + 3^2$ and this remains for all found triplets.
So all this square sums being square sums problem must be part of a larger problem of what the square sums can be at all. And they couldn't solve this problem!
But this is not the source of our second duality we mentioned above.
Instead, it starts with the also ignored a , b values where there is no triplet at all.
The simplest being (1 , 1 , c).
Could it be that these are triplets too but require smaller unit than 1?
In other words, could we have some (m , m , n) triplet with $m^2 + m^2 = 2m^2 = n^2$?

This is a sheer number theoretical impossibility! So would have fitted into the whole number approach but unfortunately was not recognized again by neither the Babylonians nor Pythagoras.
I will show the argument now.

The basic fact is that odd times odd is odd since $(2j + 1)(2k + 1) = 4jk + 2j + 2k + 1$.
This implies trivially that an even square is dividable by 4. Indeed, both members must be even.
Now comes something interesting. If we divide such even square with 4 by dividing both same members with 2 then the result is either an even square dividable by 4 again or an odd square.
And so dividing with 4 as many times as possible, we must end up with an odd square.
Now we apply 4 divisions on both sides in the assumed $2m^2 = n^2$ equality as long as possible.
We show that the last $2(m_0)^2 = (n_0)^2$ equality is visibly impossible.
The 2 multiplier on the left means that $(n_0)^2$ is even for sure and so must be dividable by 4.
Thus the dividability by 4 had to stop in m^2, that is $(m_0)^2$ is odd say $2k + 1$.
But then $2(2k + 1) = 4k + 2$ is not dividable by 4 contradicting that the other side is.

We do not know for sure but it seems that one of Pythagoras' student or cult member Hippasus did recognize all this. He was drowned for sure for revealing some secrets of the society.
As you can see from all these I am not too fond of Pythagoras and indeed it is an awful shame that this most important equation bears his name.

Euclid was well aware of the fact that the (1 , 1 , c) triangle can not be made into a perfect (m , m , n) triplet with new units. In fact, he proved this for many other cases.
He argued that since in an (m , m , n) right angle triangle n must be even thus we can cut this in half and get a new similar one. It is actually $(\frac{n}{2}, \frac{n}{2}, m)$.
Here m must be even again and so we can halve it again.
Thus we would get smaller and smaller triangles always with whole sides, that is being multiples of the original unit. But this is impossible.

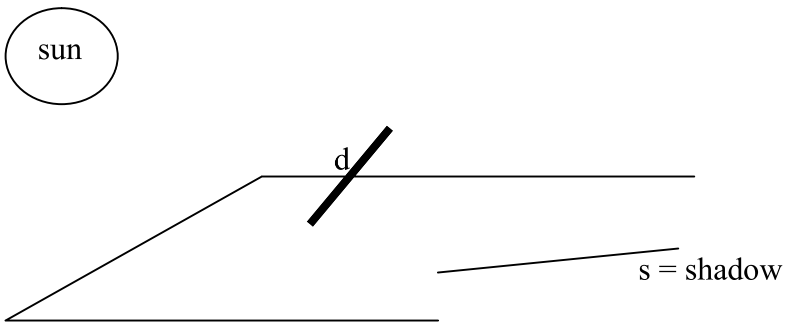
The modern solution is much nicer. We simply say that:
If there were possible $2m^2 = n^2$ equations then there had to be a smallest $2(m_0)^2 = (n_0)^2$ too.

The other side of the modern thought system is even more important.
This says that the Pythagoras Theorem is perfect so $1^2 + 1^2 = 2 = c^2$ but the c distance coming from this, which we denote today as $c = \sqrt{2}$ is an irrational number.
Irrational simply means not fractional. But what is it then? This is the missing essence.
So what numbers can describe the distances?
Well, we use the stupidest "real" numbers word which actually came from the even later imaginary number recognitions.
The correct answer is that distances are infinite decimals.
So the base ten number system has this even more important continuation that we teach already in elementary schools yet the Greeks had no vision of at all.
After the decimal point we use power fractions of ten.
The crucial elementary school algorithm is the division process.
This shows that a fraction must be a periodical infinite decimal and so $\sqrt{2}$ can not be such.
Of course, how to find this non periodical infinite decimal from 2 is not taught any more.
When I attended the math high school, my older brother at a humanity high school still did learn not only Latin but this square root algorithm. Of course we do not need this algorithm to understand that there is an infinite decimal that gives $\sqrt{2}$.

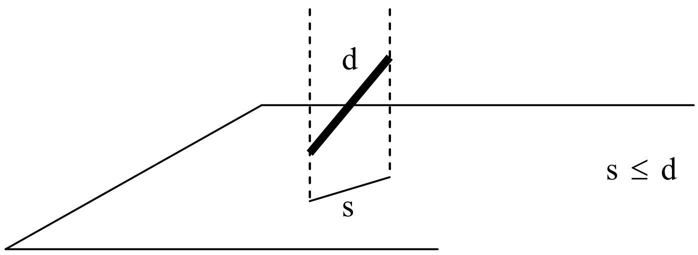
But now I leave this second detour too because a more important third duality behind the Pythagoras Theorem must be revealed.
This simply is that both the multiplication and division of the "real" numbers, that is the infinite decimals have intuitive meanings for the distances.
So here we encounter what the basic difficulty with math is! Namely, that we can use plausible arguments before even we can precisely define what we deal with.

Actually, the ratios or proportionality of distances, that is their divisions is more intuitive than their multiplications as areas. And yet the Pythagoras Theorem is the perfect example where the area plausibility is preferred as simpler. This is just pure short sightedness.
So if you only care about "simple" proofs of this famous "stuff" then go ahead and collect these areal proofs. But if you really want to see it inside Geometry as a whole then I will give you a better approach that is more general and more plausible at the same time!
So here we start with proportionality and actually not even restricted to a plane.

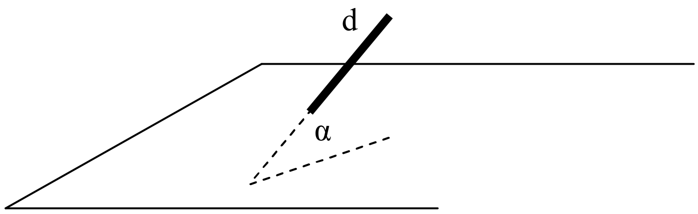
We regard a d distance and regard its shadow onto a horizontal plane:



If the sun is above around noon, then the shadow can not be longer than d.

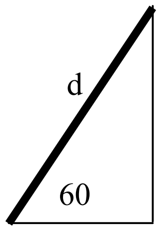


The compression ratio of s from d only depends on d's angle:



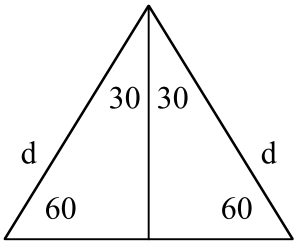
So now $s = d\,f(\alpha)$, that is d multiplied with the $f(\alpha)$ compression factor.
Observe that $0 \le f(\alpha) \le 1$.
The two extreme possibility of the f compression factor function is at once clear too.
The 0 shadow is only possible if the d distance is perpendicular to the plane, that is $\alpha = 90$.
While the f factor being 1 that is the full d becoming the s shadow happens if $\alpha = 0$.
The full details of how f changes with α is a separate matter but we can introduce a particular name already for f and it is "cosine" and abbreviated as $\cos(\alpha)$.

A particularly simple compression ratio is half and it happens at 60 degree, so: $\cos(60) = \frac{1}{2}$.



$s = \frac{d}{2}$

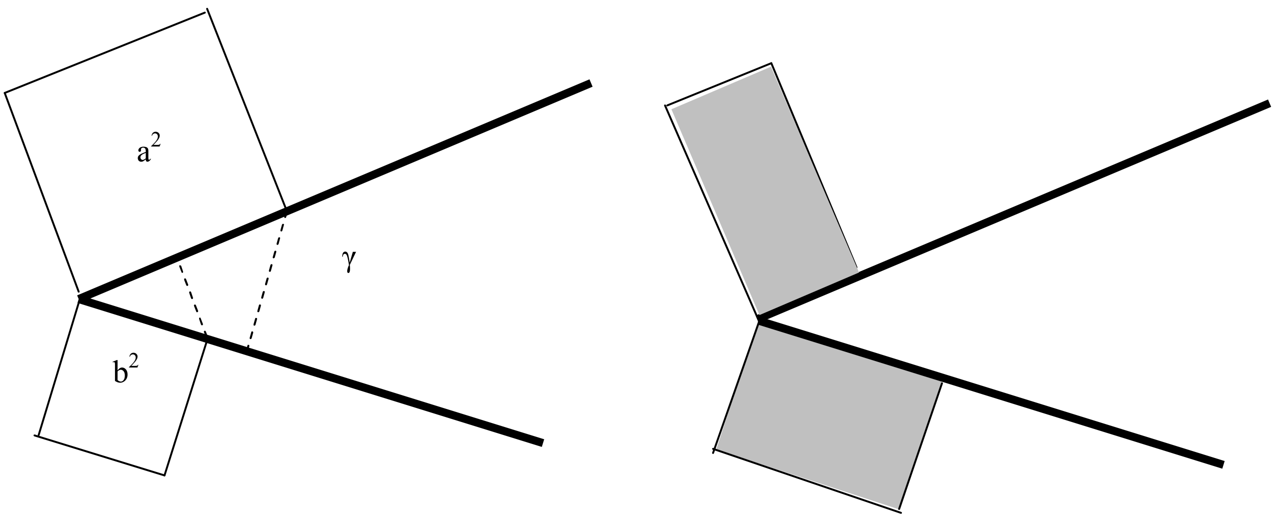
This is so, because the other angle being 30 , mirroring it will make a triangle with all angles being 60 :



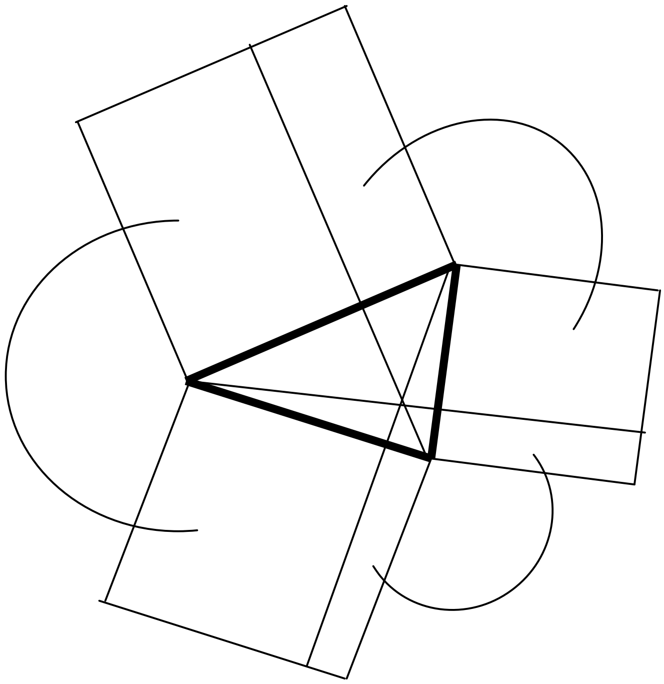
But then, since all angles are equal, the sides must be equal too, so the base is d too.

Thus, s is indeed $\frac{d}{2}$.

Now we step into a fix plane and regard a fix angle that we will denote now as γ .
The reason is that now we will regard two distances on both lines of this angle and regard both shadows on the opposite lines.
The distances will be a , b and so their angles is actually γ . Assuming that it doesn't matter in which direction we regard an angle, the two shadows are $a \cos(\gamma)$ and $b \cos(\gamma)$.
So we have four distances.
Two on one side of the γ angle a and $b \cos(\gamma)$ and two on the other, b and $a \cos(\gamma)$.
Since a product is independent of the order of the multiplied members:
$a\, b \cos(\gamma) = b\, a \cos(\gamma)$.
What could such multiplication mean for distances? The area of the rectangle with those sides.
Here indeed, we can see that the order of multiplication is irrelevant.
To make these two rectangles more visual, we can already regard the full a and b sided squares and then those rectangles are parts or extensions of the squares.
So we have two squares in any γ angle as start and then alter the squares by the shadows as base lengths but keeping the square heights:



The two rectangles are then equal in area.
This could be called as the Two Squares Lemma.
Still pretty uninteresting but if we use it for all three sides of a triangle then we got a result that can already be called as the Generalized Pythagoras Theorem.
All three squares are cut in two from the corner across and any two rectangles at a corner are equal:



To call it as Generalized Pythagoras Theorem is very granted since if $\gamma = 90$ then we at once obtain the Pythagoras Theorem.
Indeed then only c^2 is cut in two and its partial rectangles are exactly a^2 and b^2.

But we don't have to specialize γ and still get something exciting.
Namely, then the two rectangles in a^2 and b^2 are not these full squares, rather less by the rectangles at the γ angle and so:
$c^2 = (a^2 - a\, b \cos(\gamma)) + (b^2 - b\, a \cos(\gamma)) = a^2 + b^2 - 2\, ab \cos(\gamma)$.
So our Generalized Pythagoras Theorem was actually the well known cosine theorem.

Finally I will show how the Two Squares Lemma can be obtained with areas without the concept of cosine:
Let the two distances placed on the two sides of γ be again a and b and their shadows to the other side a_0 and b_0 while their perpendicularly erected square sides be a' and b'.
Trivially, the (a', b) triangle turned clockwise 90 degrees becomes the (a , b').
Now observe that the first has height b_0 if we regard as base a' , so its area is $\frac{a'b_0}{2}$.

While the second has height a_0 if we regard as base b' , so its area is $\frac{b'a_0}{2}$.

Thus $\frac{a'b_0}{2} = \frac{b'a_0}{2}$ so $a'\, b_0 = b'a_0$ so the two rectangles have same area.

By the way, the two triangles can be doubled into two parallelograms and those altered into the two rectangles by cutting of and adding triangles.
Thus we can obtain a "perfect recombination" proof.