Behind Two Theorems Of Fermat

The first of the two theorems, the “Last” or as it should be called, “Lost” theorem of Fermat, claims that $x^n + y^n = z^n$ is impossible for natural numbers if $n > 2$.
The other theorem claimed that all $4k + 1$ primes are square sums.
The first has to contain the $n > 2$ condition because for $n = 2$, that is for $x^2 + y^2 = z^2$ there are natural solutions, called as Pythagorean Triples. These were known even before Pythagoras, already by the Babylonians. The simplest is $3^2 + 4^2 = 5^2$.
The number 5 itself, is $1^2 + 2^2$ and so is itself the simplest example of the other theorem, because $5 = 4 \bullet 1 + 1$ as well. So it’s clear that the two theorems inter-relate.

The ancient Babylonian solution for the Pythagorean Triples should be approached with some detours if we want to be didactical, knowing what we know today.
The simplest idea is to merely re-write the equation as $x^2 = z^2 - y^2 = (z - y)(z + y) = a\,b$.
So, we reduced at once the question of when a square is a sum of two squares, into when a square is a product. Of course, trivially x^2 is at once a product with $a = b = x$. But this is actually an impossible solution for the original y and z. Indeed, $z - y = z + y$ implies $y = 0$.
In general, from a, b we get y, z as: $y = \dfrac{b-a}{2}$, $z = \dfrac{a+b}{2}$.
Of course, these must be naturals, which again shows why $a = b$ was wrong, giving $y = 0$.
In fact, we know exactly what a, b can be to make y, z naturals. Namely, they must have the same parity to make the fractions whole. And in reverse, such a, b will always do that.
The other condition is simply to make $b - a$ positive, which is $a < b$.
Of course, $x^2 = a\,b$ with $a < b$ actually means $a < x$ and $b > x$.
Indeed, $a^2 = a\,a < a\,b = x^2 \rightarrow a < x$. Similarly, $b^2 = b\,b > a\,b = x^2 \rightarrow b > x$.
So to find the possible solutions of a, b and y, z for a given x is very simple:
We try out all $a = 1, 2, \ldots, x - 1$ values, whether they divide x^2, and if so, whether dividing x^2 with a and thus, obtaining b, we’ll get numbers with the same parity.

$x = 1$ has no smaller values, so no a exists already.

$x = 2$ allows $a = 1$ and $x^2 = 2^2 = 4$ divided by 1 gives $b = 4$.
Since a, b have opposite parity, there is no solution again.

$x = 3$ allows $a = 1, 2$ and only one divides the square $3^2 = 9$ and $b = 9$ has same parity.
So we got our first solution, $3^2 = 9 = 1 \bullet 9 = a\,b$.
So $y = \dfrac{b-a}{2} = \dfrac{9-1}{2} = 4$ and $z = \dfrac{a+b}{2} = \dfrac{1+9}{2} = 5$.
So we indeed obtained the simplest Pythagorean Triple $3^2 + 4^2 = 5^2$.

$x = 4$ means $a = 1, 2, 3$ and $4^2 = 16$ divided by 1 gives even, so is not good.
Dividing by 2 we get a good 8 as b. And finally, 3 doesn’t divide 16 at all.
Hooray! We got our second solution, $a = 2, b = 8$ so $y = \dfrac{8-2}{2} = 3$, $z = \dfrac{2+8}{2} = 5$.
So, we merely got the previous $3^2 + 4^2 = 5^2$ solution as $4^2 + 3^2 = 5^2$.
No problem! We didn’t say we can’t get such repetitions, we merely wanted all solutions.

$x = 5$ means $a = 1, 2, 3, 4$. But 25 is only dividable by the first.
So $a = 1, b = 25$, $y = 12$, $z = 13$. This is our third or second new triple: $5^2 + 12^2 = 13^2$

$x = 6$ means $a = 1, 2, 3, 4, 5$. All of them divide 36 but only 2 gives same parity b.
So $a = 2$, $b = 18$, $y = 8$, $z = 10$. So, our third new solution is: $6^2 + 8^2 = 10^2$.
Amazingly, this became smaller than the previous.

To avoid repetitions is also easily possible. All we need is that $x < y$ is guaranteed, that is:

$$x < \frac{b-a}{2} = \frac{\frac{x^2}{a} - a}{2} \quad \text{or} \quad 2x < \frac{x^2}{a} - a \quad \text{or} \quad \text{multiplying with } a \text{ and adding } a^2:$$

$$a^2 + 2ax = (a+x)^2 - x^2 < x^2 \quad \text{so} \quad a < \sqrt{2x^2} - x = x(\sqrt{2} - 1).$$

So for our next, $x = 7$ for example, $a < \sqrt{2 \bullet 7^2} - 7 = \sqrt{98} - 7 = 9. . . - 7 = 2. . . .$
So we only have to try $a = 1, 2$. Only the first divides 49 giving $b = 49$, $y = 24$, $z = 25$.
And indeed, $7^2 + 24^2 = 25^2$.
At the next $x = 8$ the new limit is $a < \sqrt{2 \bullet 64} - 8 = \sqrt{128} - 8 = 11. . . . - 8 = 3. . . .$
So $a = 1, 2, 3$. Only 2 works giving $b = 32$, $y = 15$, $z = 17$. Indeed, $8^2 + 15^2 = 17^2$.

It all works like a charm and the Babylonians would have been very happy to see all this.
But our better vision is only a potentiality. There is a dark tendency called Formalism that tries to hide exactly what we could see today. Already the expression "Pythagorean Triples" is a distortion. The Babylonians were more interested in the problem, what squares can be square sums, that is what are the z values. Our method didn't really tell this. The z values came all out, but not directly as the Babylonians wanted. They wanted to calculate z with mathematical operations. Or as we would say it today "explicitly".
Well, we could say, again with our new notations, that $z = \sqrt{x^2 + y^2}$, there you go.
But this formula doesn't always give a whole number. So, we avoided the very problem.
Yet, it was useful to toss up this crazy "solution", because the one we want is similar, in the sense that it also has to calculate z from two values, say u and v. But we want to get a correct and whole z from all possible usages of u and v. With this precise restriction, it is plausible that we need two such variables or as they are called, parameters. Indeed, one single p parameter doesn't feel to be enough to calculate such complex set of z values. Trying out any complicated $F(p)$ formulas using addition, multiplication or even exponentiation, could hardly produce all possible z values as: $F(1), F(2), F(3), \ldots$
Surprisingly, there is such $F(p)$ formula, but this is visible only by truly jumping thousands of years ahead of the Babylonians. One thing is obvious, the more variables or parameters we allow, the easier should be to find such F formula. The usage of two u, v that is searching for $z = F(u, v)$ is the historical compromise. It already allows an amazingly simple F.
The above used trick $z^2 - y^2 = (z - y)(z + y)$ raises the idea of using the two companion formulas: $(u + v)^2 = u^2 + v^2 + 2uv$ and $(v - u)^2 = v^2 + u^2 - 2uv$.
Since they have the same members, except the first has $+ 2uv$, while the other $- 2uv$, thus, adding them together would cancel this member. But, if we instead subtract the second from the first, then we get merely $4uv$. What's the point of this, you may ask? Well, the number 4 is a square, so if instead of u, v we use u^2 and v^2 then $4u^2v^2$ is a perfect square of $2uv$.
This of course means using u^2 and v^2 in the sum and difference square formulas too and so we get the following equality: $(u^2 + v^2)^2 = (v^2 - u^2)^2 + (2uv)^2$
This proves at once that $z = F(u, v) = u^2 + v^2$ is a sufficient formula with the added assumption of u and v being different numbers. Indeed, then we can assume $u < v$ and so $x = v^2 - u^2$ is a natural number and $y = 2vu$ is the other. Their square sum is z^2 by the above equality.
Is this square sumness of z necessary too? That is, only such z can have a square that is a square sum? Clearly not, as $15^2 = 9^2 + 12^2$ shows. But this counter example is strange because both 9 and 12 have the factor 3 and thus $9^2 + 12^2$ could be written as $3(3^2 + 4^2)$.

In short, this $9^2 + 12^2$ square sum can be simplified to the $3^2 + 4^2$ case.
A simple solution is one that can not be simplified anymore. In other words, where x and y have no common factor or as it is called, they are relative primes.
These relative prime or simple solutions were mostly what the Babylonians wanted to produce with a single $F(u, v)$ formula.
And what do the concrete examples tell? Well lets see the simple square sums:

$$3^2 + 4^2 = 5^2 \;,\; 5^2 + 12^2 = 13^2 \;,\; 7^2 + 24^2 = 25^2 \;,\; 8^2 + 15^2 = 17^2 \;,\; \ldots$$

Quite amazingly, all these have z values that are square sums themselves:

$$5 = 1^2 + 2^2 \;,\; 13 = 2^2 + 3^2 \;,\; 25 = 3^2 + 4^2 \;,\; 17 = 1^2 + 4^2 \;,\; \ldots$$

But why? And will this continue without exemption? The Babylonians couldn't answer these questions precisely, but they did realize the formula: $z = F(u, v) = u^2 + v^2$ and also realized the equality we used above. Thus not quite precisely but they actually proved that

$$z = u^2 + v^2 \text{ and } u \neq v \qquad \rightarrow \qquad z^2 = x^2 + y^2$$

A simple reversal of this, that is:

$$z^2 = x^2 + y^2 \qquad \rightarrow \qquad z = u^2 + v^2$$

is not true as the already mentioned $15^2 = 9^2 + 12^2$ example shows. But it is true that:

$$z^2 = x^2 + y^2 \text{ and } x, y \text{ are rel. primes} \quad \rightarrow \qquad z = u^2 + v^2$$

The Babylonians knew this too from the examples but didn't prove it at all!
To show it today, strangely we again need merely two formulas:

$$u = \sqrt{\frac{z-x}{2}} \quad , \quad v = \sqrt{\frac{z+x}{2}}$$

First observe that at once, $u^2 + v^2 = \frac{z-x}{2} + \frac{z+x}{2} = z.$
This came out so easily, that we feel something being "fishy". Indeed, we obtained z, yet we didn't use the very condition $z^2 = x^2 + y^2$ nor x, y being relative primes.
We might even think that our formulas are wrong. They are perfectly correct.
But u, v are given as square roots. So it is not obvious at all, that these are whole numbers.
To prove this we have to use the condition $z^2 = x^2 + y^2$ and x, y being relative primes.
There are two obvious steps! First, why the fractions under the square roots are wholes.
Second, why they must be square numbers, so the square root gives wholes too.
For these fractions to be wholes, means that $z - x$ and $z + x$ are even.
To see this, we will show some wider facts first.
Above we said that the Babylonians were looking for simple solutions where x and y have no common factors. This actually means that none of x, y, z can have common factor.
In fact, this is true for the general $x^n + y^n = z^n$ equality.
Indeed, if any two from x, y, z have a p common prime factor then the third must have it too. This is so because being factor and the commonness of p, remains for the powers of the two and thus also implies being factor of the power of the third because it is sum or difference of the other two. Finally, since p is prime, if it is factor of a power then it is factor of the base too.

In particular for 2 factors or evenness too, already two can not be such.
So maximum one of x, y, z can be even. But a different argument shows that one must be:
Indeed, all odd x, y, z numbers would mean all odd powers too and so $x^n + y^n$ were even while the other side z^n is odd. So quite generally, exactly one of x, y, z is even.
Now returning to the special $x^2 + y^2 = z^2$ Pythagorean case, here the even can not be z.
Indeed, the square of an even number is $(2k)^2 = 4k^2$ and thus, is dividable by 4.
Now the other side would have the sum of the two odd squares. But:
The square of an odd number is $(2k+1)^2 = 4k^2 + 4k + 1$ and thus, has 1 remainder to 4.
So, the sum of two such numbers must have 2 remainder to 4 and thus, could not be equal to the other side, which as we showed, was dividable by 4.
So, z must be odd and one of x or y is odd too. Their role is symmetrical, so we can choose y to be the even. Then, z and x are the odds and so, indeed, $z - x$ and $z + x$ are evens, so the fractions in our formulas give whole numbers. But why are they squares? This is a bit trickier:
First observe, that $\frac{z-x}{2}$ and $\frac{z+x}{2}$ are relative primes too. Indeed, if they had a c common factor, then this c would divide their sums and differences, which happen to be z and x.
But of course, we assumed that these are relative primes. So the two fractions are relative primes.
Now comes the real trick, lets multiply these together:

$$\frac{z-x}{2} \bullet \frac{z+x}{2} = \frac{(z-x)(z+x)}{4} = \frac{z^2-x^2}{4} = \frac{y^2}{4} = \left(\frac{y}{2}\right)^2$$

We showed already that y was the even, so $\frac{y}{2}$ is a whole and thus, the product of the two fractions is a square for sure. But why are they squares themselves?
Simply, because they are relative primes. The $\frac{y}{2}$ number can be anything, having any kind of prime factors, like $2 \bullet 3 \bullet 3 \bullet 5 \bullet 13$. But, the square of this will definitely mean doubling all these prime factors. Now, since the two fractions multiplied is equal to this square, it must have double occurrences of all prime numbers. But also, them being relative primes means they don’t share any prime factors. So indeed, both of them contain doubles of different prime factors. So, both of them are squares. Thus, we proved that our formulas for the “reverse” of Babylon are wholes.
I am not going into now why the Babylonians couldn’t find these reverse formulas.
What is more important is that you can go on the net today, type in Pythagorean Triples and you’ll find a flood of overcomplicated and stupid abstractions except the formulas I showed.
This is that should concern you and if it’s not then you are in deep trouble yourself!

Returning to our subject. There were two points not quite precise in our arguments above.
The first, an easy acceptance of that the non simple or expanded solutions of $x^2 + y^2 = z^2$ can always be reduced to simple ones, the other the usage of the prime factorizations.
These two actually inter-relate in an interesting way.
We might say that the simplification process speaks for itself, because whenever there is a c common factor, we just divide with it and thus, the values decrease, so eventually, we must obtain a simple or relative prime solution. The really puzzling question is whether the end result, depends on how we made the lucky finds of c common factors. To see that luck has nothing to do with it, that is the end result is always the same, is easy by looking at the x, y, z numbers as each being products of primes. Then, a c common factor is merely common primes with maybe repetitions, but this repetition can not exceed the repetitions we have in either of the numbers.
So indeed, doing the full simplification means picking all the common prime factors with the highest possible repetitions allowed by the numbers. And thus, this does not depend on how we go, it is determined by the x, y, z numbers themselves.
To see that all numbers have prime factors, can go again by repeating the same argument, that is arbitrary divisions. We try to divide a number with anything possible, making it a product, then break down the two factors again into smaller factors and so on.

The big difference, using one number alone, is that here the break down only stops when we reach factors that are not dividable at all, rather are so called primes. But the main question remains the same. Why is the order of this break down irrelevant? Why do we get a Unique Prime Factorization?
Strangely, the easiest proof of this Unique Prime Factorization of individual numbers relies on the concept of relative primes. Namely, on the fact that if a d divider of an $A\,B$ product is relative prime to one of them, say A, then d must divide B.
Indeed then, for $d = p$ prime, it follows that it must divide at least one of A , B:
If p doesn't divide A, then this already means that p and A are relative primes, because p doesn't have factors at all. So, p must divide B.
This then generalizes to products having more members and in particular to $q_1\ q_2 \cdot \cdot \cdot q_n$ products of primes too. But here, p dividing one of the members, say q_i means $p = q_i$ because primes have no factors.
This shows that a $p_1\ p_2 \cdot \cdot \cdot p_m = q_1\ q_2 \cdot \cdot \cdot q_n$ equality can only be if $m = n$ and the members are the same too on both sides, except maybe in different order. Indeed:
We can regard p_1 as p which divides the other side, so it is a q_i. Then, we can divide with this and regard p_2 as p. And so on, gradually the two sides disappear, proving what we claimed.

Returning to Pythagorean Triples, in spite of the Babylonians realizing that $z = u^2 + v^2$ was the condition of $z^2 = x^2 + y^2$, this wider question of when numbers themselves are square sums was strangely never raised. It is strange because this simpler question is much deeper.
The fundamental start is the recognition that two square sums multiplied is again a square sum:

$$(a^2 + b^2)(c^2 + d^2) = a^2 c^2 + a^2 d^2 + b^2 c^2 + b^2 d^2$$

From this it's not clear yet at all why this should be a square sum.
The trick is to add and subtract $2\,a\,b\,c\,d$, and so:

$$(a^2 + b^2)(c^2 + d^2) = a^2 c^2 + b^2 d^2 + 2\,a\,b\,c\,d \ + \ a^2 d^2 + b^2 c^2 - 2\,a\,b\,c\,d$$

But now the first three are a square, and the second three as well. So:

$$(a^2 + b^2)(c^2 + d^2) = \quad (a\,c + b\,d)^2 \qquad + \qquad (a\,d - b\,c)^2 .$$

Observe the rule:

$$\begin{tikzcd} a \arrow[d] & b \arrow[d] \\ c & d \end{tikzcd} \qquad\qquad \begin{tikzcd} a \arrow[dr] & b \arrow[dl] \\ c & d \end{tikzcd}$$

$$+ \qquad\qquad\qquad\qquad -$$

This seemingly obscure trick and the resulting multiplication formula has an even more puzzling "behind picture", if we use the $\sqrt{-1}$ "imaginary" square root as a distinguisher of the second members in our square sums. So, $x + y\sqrt{-1}$ is a combined sum of a real and imaginary number. But, as it turns out, these sums produce all the necessary tricks of square sums.
To see this, first we have to realize that the square sums themselves become products, with the help of these sums. Indeed, the old $(a + b)(a - b) = a^2 - a\,b + b\,a - b^2 = a^2 - b^2$ trick that produced only differences of squares, now will give sums of them:

$$(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 \cancel{- a\,b\sqrt{-1}} \cancel{+ b\sqrt{-1}\,a} - b^2\left(\sqrt{-1}\right)^2 = a^2 - b^2(-1) = a^2 + b^2$$

Now, our original $(a^2 + b^2)(c^2 + d^2)$ product can be regarded this way too, thus:

$$(a^2 + b^2)(c^2 + d^2) = (a + b\sqrt{-1})(a - b\sqrt{-1})(c + d\sqrt{-1})(c - d\sqrt{-1}) =$$

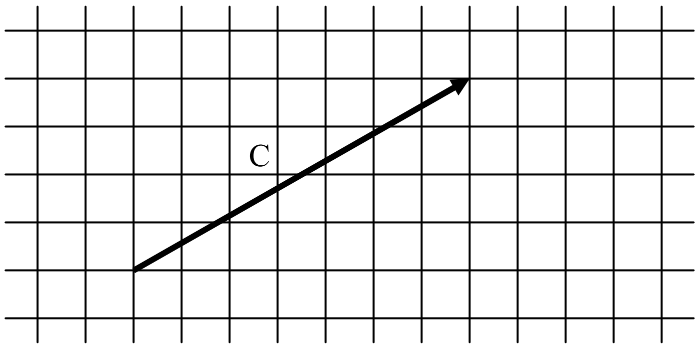
$$(a + b\sqrt{-1})(c - d\sqrt{-1})(a - b\sqrt{-1})(c + d\sqrt{-1}) =$$

$$\left(a\,c - a\,d\,\sqrt{-1} + b\,c\,\sqrt{-1} - b\,d\,(-1)\right)\left(a\,c + a\,d\,\sqrt{-1} - b\,c\,\sqrt{-1} - b\,d(-1)\right) =$$

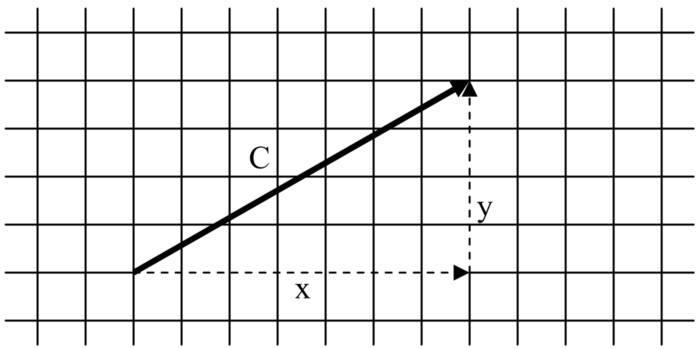
$$\left((a\,c + b\,d) + (b\,c - a\,d)\sqrt{-1}\right)\left((a\,c + b\,d) - (b\,c - a\,d)\sqrt{-1}\right) =$$

$$(a\,c + b\,d)^2 - (b\,c - a\,d)^2\,(-1) = (a\,c + b\,d)^2 + (b\,c - a\,d)^2 .$$

So we obtained exactly our result above, without using the trick of adding and subtracting members, rather apply $\sqrt{-1}$ as a temporary trick. This trick was much longer and tedious, but it produced the result, completely automatically, using only the laws of algebra. In a sense, this long and tedious solution to replace the original, innovative trick could be done by a computer.
This was the "vision" up until Gauss finally saw behind the real role of $\sqrt{-1}$ being not just some "automated" trick eliminator or producer, rather a meaningful reality. He wasn't keen to explain what he saw, rather turned his vision to create more proofs. Exact proofs, without the grand new picture. No wonder they called him the "fox", that also erases its tracks with its tail.
I would like to reveal this grand vision now:
So imagine an infinite bathroom tiling with simple square tiles.
No fashionable pictures on the tiles, no coordinates in the plane either, just squares.
The "connectors" are arrows from any corners to any other in this tiling:

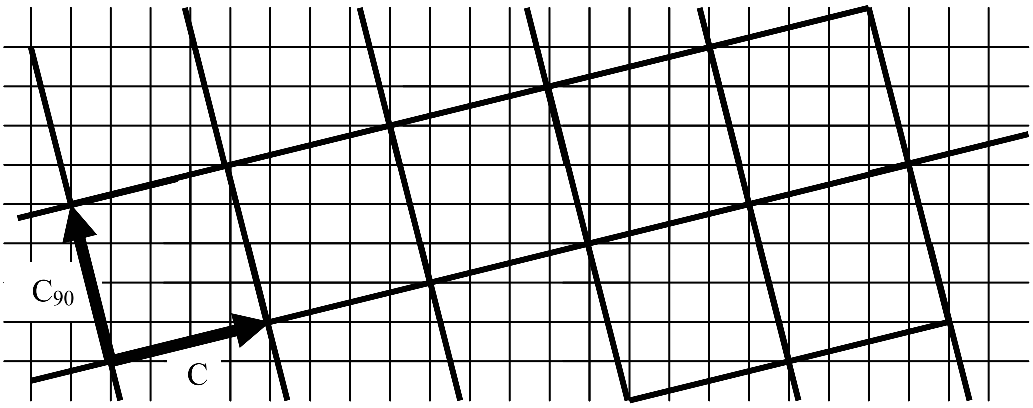


These C connectors are our new numbers that will represent the $x + y\sqrt{-1}$ sums.
As we can guess from this, x will mean the horizontal and y, the vertical steps that achieve the same motion as C itself:



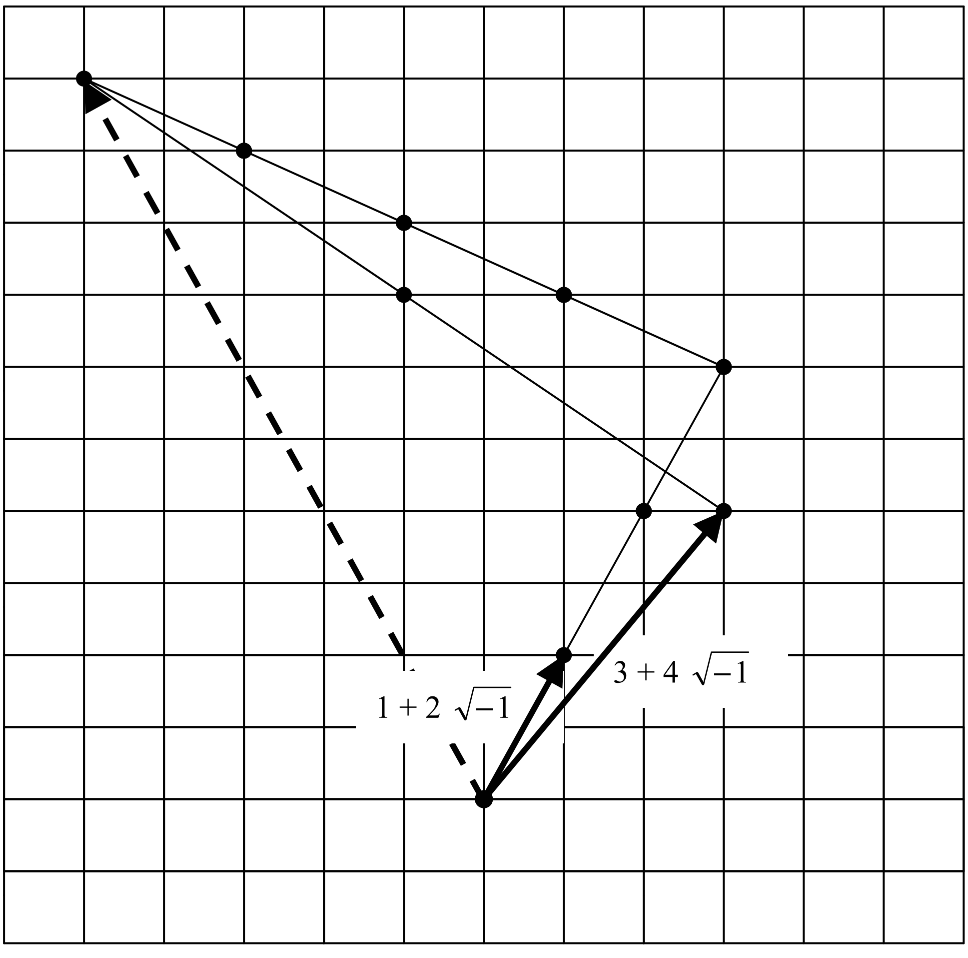
Of course, x , y can be zero or negative whole numbers too.
Zero means not moving horizontally or vertically at all, and minus means moving left or down.
This acceptance of the right and up as plus is merely our human convention.
So, the combined or complex number $(-2 + 3\sqrt{-1})$ for example means:
Go 2 tiles horizontally to the left and then go three tiles vertically up.
As an other example : $(7 - 2\sqrt{-1})$ means go 7 right and 2 down.

The crucial first recognition about our new universe is this:
Our tiling has hidden sub-tilings in it. Namely, every fix C connector generates one.



The repeated continuations of C in its line, falling onto exact corner points, that is being connectors, is obvious. The less obvious fact is that turning C with 90 degrees, this C_{90} will be a connector again. Of course, it follows from turning both of the x , y components of C too. Then, the repeated continuations of C_{90} give the other perpendicular sides of the new tiles.
This vision is elementary school stuff. Every kid should see it as early as possible.

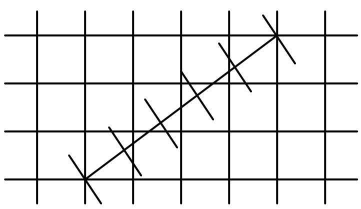
The second important vision is using any U connector as new unit or fixed sub-tiling and to measure a second C connector in this system. The surprising fact is the following:
C measured in the U sub-system leads to the same point as U measured in C. For example:



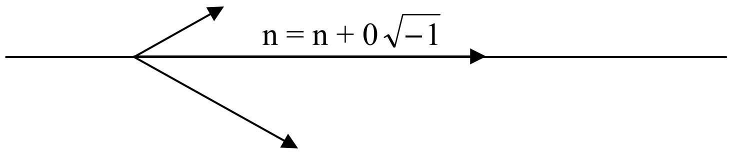
The two connectors are the black arrows: $1 + 2\sqrt{-1}$ and $3 + 4\sqrt{-1}$.

The resulting connector is the dashed arrow and easy to see through the black dots we placed, that it is both $1 + 2\sqrt{-1}$ measured in the $3 + 4\sqrt{-1}$ unit system or in reverse.
This common end result is quite surprising as a general fact. Yet trying every concrete example, it turns out to be true.
One "ugly" way to prove this sameness is to realize that the representations of the connectors in each other's sub-system is actually the product of them, using the $\sqrt{-1}$ compositions.
For example, above, the resulting vector is $-5 + 10\sqrt{-1}$. And voila:
$(1 + 2\sqrt{-1})(3 + 4\sqrt{-1}) = 3 + 4\sqrt{-1} + 6\sqrt{-1} + 8(-1) = -5 + 10\sqrt{-1}$
The same can be seen in general.
But there is a nicer way to see this fact by reinterpreting the multiplication itself.
We have to regard connectors determined in an other way than from the x and y components.
Namely from their length $|C|$ and their angle to the horizon.
Then, using U as unit, means replacing the x , y , C triangle with a similar one, but using as unit, not the original tiles rather new $|U|$ tiles. So this means a $|U|$ stretching of $|C|$ and a turn of its angle with U's angle.
So the representation of C in U will have $|U|\,|C|$ length and the sum of their angles,
This at once shows that the order is immaterial and U's representation in C is the same.

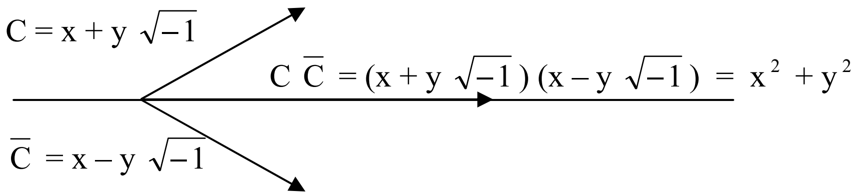
The third crucial thing to "see" is what we can't really see, namely that the square sums are the squares of these $|C|$ connector lengths, that is $|C|^2$.
Indeed, $x^2 + y^2$ of a C connector is obviously the square of its length, that is, $|C|^2$ by Pythagoras theorem. But the $|C|$ length is all we can see. Of course, we could place a square on top of C but that wouldn't fit into our tiling at all. So we simply have to accept that $|C|$ is not a whole number, but $|C|^2$ is. Yet the $|CU| = |C|\,|U|$ rule implies $|C|^2 |U|^2 = |CU|^2$ too.
So this gives a new proof of that square sums multiplied remain square sum.
A result for whole numbers thus obtained temporarily stepping out to the real numbers.
Indeed, $|C|$, $|U|$, $|CU|$ are not wholes but produce the product as square sum.
Observe the concrete example: $(1^2 + 2^2)(3^2 + 4^2) = |1 + 2\sqrt{-1}|^2 \, |3 + 4\sqrt{-1}|^2 =$
$= |(1 + 2\sqrt{-1})(3 + 4\sqrt{-1})|^2 = |-5 + 10\sqrt{-1}|^2 = 5^2 + 10^2$
The $\sqrt{-1}$ meant the vertical component and so repeated application should mean 180^o turn.
On the other hand, we know that $\sqrt{-1}\,\sqrt{-1} = -1$. These two mean the same thing.
Indeed, 180^o turn is actually multiplication with -1.
So $\sqrt{-1}$ is not imaginary at all, it is the half turn of negativity or 180^o turn, that is a 90^o turn.
These $x + y\sqrt{-1}$ complex numbers used with x , y real numbers instead of wholes, became the real language of modern physics. Gauss had this vision, but couldn't foresee the electromagnetic fields, quantum mechanics or relativity yet. This was the root of his bitterness.
What is important from our subject is that these complex numbers of the full plane, should be approached through the tilings as new whole numbers first. Just as the real numbers must be approached through the naturals and fractions.
Back to our subject, when $x^2 + y^2 = z^2$ is true with a whole z, that is the Pythagorean Triples, simply mean a C connector that $|C| = z$. For these connectors, quite accidentally, our tiles could be placed to cover the z length. The simplest case of course is $3^2 + 4^2 = 5^2$.
So this is also a certain "sub-tiling", but it has no meaning to be used further.



This fact that already the Pythagorean Triples don't fit into the tiling system or Gaussian integers, is a sign that probably Fermat's Last Theorem also has nothing to do with this vision.
Even more evidence of this will be shown.
The final, fourth crucially visible fact that lead to the understanding of square sums is that the horizontal connectors as old fashioned natural numbers, can be the products of two connectors, only if these two are symmetrically angled to the horizontal.

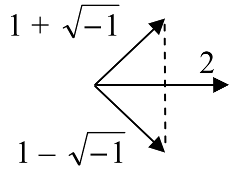


This fact is true in general, for the complex or plane numbers. The horizontal old fashioned x real numbers can only be products of symmetrical pairs. Indeed this follows from multiplication adding the angles. The horizontal 0 angle can be the sum of only α and $-\alpha$.
Equal long, symmetrical complex factors thus must be $x + y\sqrt{-1}$ and $x - y\sqrt{-1}$ so called conjugates and these are indeed the crucial in all physical applications to bridge the abstract complex numbers to the measurements as real numbers.
Among the integers or connectors, these C, $\overline{C}$ conjugates give the square sums:

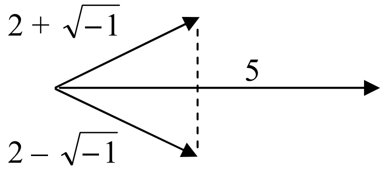


Observe that this fourth vision simply goes back to the algebraic start.
This is how we introduced the imaginary members to make a square sum a product.
The simple square sums, that is when x, y are relative primes, means that the $C = x + y\sqrt{-1}$ and $\overline{C} = x - y\sqrt{-1}$ connectors are simple too, namely that there are no grid points on them.
We might jump to the conclusion that these simple or minimal C connectors are then the prime numbers of the plane. But this is not true!
On the line, among the naturals, we only have numbers and primes. In the plane, these minimal connectors are an in between class. All primes must be such, but not all of them are primes.
In spite of this, these minimal connectors are crucial in showing how the old numbers fit into the new. The fact we saw so clearly that square sums multiplied give square sum, is only reversible among the simple square sums. Simple square sums have same kind of factors.
This fact comes out only through this fourth mirroring vision.
The best use of the mirroring vision is that the particularity of why "half" of the old primes are square sums, like: $2 = 1^2 + 1^2$, $5 = 1^2 + 2^2$, $13 = 2^2 + 3^2$, $17 = 1^2 + 4^2$, . . .
are not merely strange empirical facts now. Indeed, these are the old primes that are not primes in the plane anymore, rather products of two symmetrical primes of the plane:

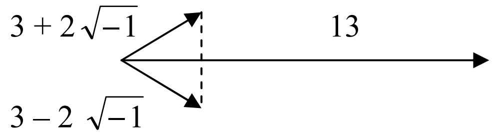
$$2 = (1 + \sqrt{-1})(1 - \sqrt{-1}) = 1^2 - (\sqrt{-1})^2 = 1 + 1$$



$$5 = (2 + \sqrt{-1})(2 - \sqrt{-1}) = 2^2 - (\sqrt{-1})^2 = 4 + 1$$



$13 = (3 + 2\sqrt{-1})\,(3 - 2\sqrt{-1}) = 3^2 - (2\sqrt{-1})^2 = 9 + 4$



In the last two we could have used the "opposite" pairs $1 + 2\sqrt{-1}$ and $1 - 2\sqrt{-1}$ for 5 and $2 + 3\sqrt{-1}$ and $2 - 3\sqrt{-1}$ for 13. But these are not new primes!
Indeed, 1, -1, $\sqrt{-1}$ and $-\sqrt{-1}$ are the four units of the plane numbers. So multiplying with these counts as same numbers or merely unit variants.
So, for example: $1 + 2\sqrt{-1} = \sqrt{-1}\,(2 - \sqrt{-1})$ and $1 - 2\sqrt{-1} = -\sqrt{-1}\,(2 + \sqrt{-1})$.
This new insight into the square sumness of the 2 , 5 , 13 , . . . primes, still leaves the remaining 3 , 7 , 11 , . . . ones empirical. Why are these remaining primes as connectors too?
Surprisingly, there was an old rule relating to this square sum split of the primes.
This is exactly the second theorem claimed by Fermat. The amazing coincidence that:
The $4k + 1$ primes are square sums and the $4k - 1$ are never. Only the first part is hard.
Unfortunately, this does not come out from the plane picture at all.
It merely boils down to that for all $p = 4k + 1$ primes we can create an artificial and fairly big simple square sum that contains p as prime factor. Then the inheritance of simple square sumness to factors implies that p is such of course. So the square sumness of p is not inside p, rather outside among the larger numbers that contain p. In general too, properties of numbers are not merely inside of the numbers because these properties are defined with all naturals in sight.

The highest mystery of numbers is where we started from, the infamous Fermat's Last Theorem.
That for naturals, $x^n + y^n = z^n$ is impossible if $n > 2$.
Finally we have a proof, but it is a whole book, and goes away, far from the naturals to come back with merely the result. No explanation, why is this so? Is this the future? The world is complex and abstract and we ran out of plausible explanations? I don't think so. I think we are merely in a ditch. Namely our "blind spots" got condensed.
But the "why" of this theorem is not the only mystery, the "how" is too. And here we have details. In fact this "how" splits in two. The strange details of the little we know and the even more strange details of how this little is distorted and covered up by Formalism.
I became completely convinced through my involvement with New Math that the existence of this menace Formalism, the systematic and organized lie is related to the blind spots.
The mania that proof is all that counts, is the limitation of us finding new proofs.
When I got involved with these older results, I was amazed how my belief in New Math's disease is true even more for the "good old math".
The only thing I can do is to uncover some lies. To show the pictures beyond the formal proofs.
Some final words about the "dark side".
This human role of Formalism which is simple lying is not a new thing. Our age spreads the lies with higher speed but the lying itself is old. Vanity and fame has been ruling science publications for ever. Fermat the judge, knew exactly well that he did not have a proof for his Last Theorem and merely wanted to muddy the waters, so his priority is secured. In his letters to friends he continually lied about having proofs for his claims that are "too long to be included". Euler did not use the same strategy but still regarded publishing half baked proofs more important than to wait and dig deeper. Finally, Gauss realized that faults in the proofs will be always picked up by some, so he had to be impeccable in this respect. But this new rigor was ego motivated. His pretentious devaluation of Fermat's Last Theorem was a strategy too. If someone else solves it then he can still look good. He knew it exactly well that this was the most important unsolved problem, just as it is today even after a proof. Lying about our opinions is okay as long as our proofs are perfect. Explaining or giving behind vision was completely avoided as if it were weakening the strictness, the high stand. The tragedy is that Academia, the parrots copied this attitude, in fact, strived to over do the creators in exactness and abstraction. The masters not revealing their opinions, their behind the proof insights became the perfect cover for the parrots

who don't even have a vision and useful opinion. The new creative pupils had to dig out their visions themselves and learnt never to give this away either. This is the re-generation of Formalism. By today we are dealing with a global conspiracy.
The new counterpart of Academia, Media is the silent partner of this conspiracy. Their trivialities are in effect the mystification of Academia as the official yardstick of truth. Indeed, if the insane common sense of Media is a lie, then common sense can not be the source of truth. The feeling that somebody up there knows all, is the total social mind control of the individual. The individual is too weak to start to think and falls back to consume and being entertained. The individual nevertheless feels this trap and wants to break out. Especially the young who always enters into a generational conflict, falsely identifying the root of the lies. The breaking out has only one outlet, to become privileged, to become some level of puppeteer and thus forget the acceptance of stupidity. This brings about undeserved consumption, getting more than the privileged produces. This is softened by rationalizing its own role but never resolved. This brings about guilt and protecting the whole social system. The conspiracy becomes taboo.

Fermat only proved his grand claim for $n = 4$. That is, being $x^4 + y^4 = z^4$ impossible.
His method was the so called "infinite descent".
Euler was the first to prove the much harder $n = 3$ case but his proof contained a serious gap.
There are voices claiming that his other publications contain the missing parts, so in effect he proved the case. This is a complete misrepresentation of the truth. The gap is the whole essence of the proof. It is the crucial special feature that relates the forms like square sums to the claim of Fermat's Last theorem. An other interesting side of the story is that initially Euler was very critical of Fermat and yet he used the same method, infinite descent.
So, lets clear first this method.
The misconception is that it is only usable for proving infinite impossibilities.
But such impossibility merely means that for all numbers something is not true.
Which means that the negative of the something is true. So what's the big deal?
To prove that something is true for all numbers is usually done with induction.
We show it for 1 and then prove that it inherits from n to $n + 1$. End of story.
Well, the first complication is if we deal with not a property about a single n number rather a relation of numbers. In our case this is the case even for a fix power say 4. The relation is that $x^4 + y^4 = z^4$ is not true. The initial 1 case is $x = y = z =1$ and indeed we can see it is true because $1 + 1 = 1$ is false. The stepping to new cases is problematic though. We can't force to step all three variables in synchrony. It wouldn't cover all number combinations.
But this is not a deep problem! Already the singular n induction has a more flexible version when we don't step from n to $n + 1$ rather from all $n < N$ to N.
Indeed, if the truths of all $n < N$ cases imply the same for N, we covered all cases.
This then at once is usable for more numbers.
For example if $u^4 + v^4 = w^4$ being never true for u, v, w all being strictly under N, imply that $x^4 + y^4 = z^4$ is not true either with one of these being N, then we are finished.
Indeed, all values of x, y, z have a maximal among them and this as N was achieved from smaller cases.
A negative version of induction is the minimality principle. Here we regard not R.
To show that R is always true means to show that there are no not R cases that is counter examples for R. If there were then there would have to be a "first" or rather one with minimal N value. The induction step actually proves that the under non counter examples imply the up to ones to be non counter example too. So this first counter example can not exist.
The initial condition was still crucial here. There are no $u, v, w, \ldots$ values under 1 and so the fact that $1, 1, 1, \ldots$ is a non counter example only comes from this.
However, the induction step can hide the initial condition. Namely, if we can prove that the under N $u, v, w, \ldots$ values imply the up to N $x, y, z, \ldots$ values without an assumed existence of $u, v, w, \ldots$ then actually we proved the initial case.
This can be viewed in the negative minimality principle with using negativity further inside the induction step. This step is an implication of universalities. All under cases imply all up to cases.

A implies B is the same as not B implies not A.
And not all u, v, w, . . . A means : there are u, v, w, . . . not A. Similarly for x, y, z, . . .
So, if there are no existences assumed about u, v, w, . . . then the induction step simply claims that an x, y, z, . . . up to N counter example implies a same u, v, w, . . . under N one.
And indeed, we can see directly without induction too why this will mean the universality of R.
Namely, if there were any counter example then we could always guarantee an earlier one.
But descending numbers can not descend infinitely. So this is infinite descent.
And this can be "positive" too. As an example, regard the claim:
All composite numbers have prime factor.
Usually this is proved by visualizing repeated divisions that must end in primes.
Infinite descent gives the same result instantly.
Indeed, c being composite means being a b with none of them 1 or itself c. Having no prime factors implies that these are composites too. So they are smaller composites. But they can not have prime factors either, otherwise c had them too.
So the property that compositeness implies prime factor is such that its negative, that is being composite without prime factor, implies smaller similar one. Thus, all numbers must have the positive property, that is if they are composite they have prime factor.
Here, the negativity, having no prime factor was the inheriting to smaller cases.
A further step in making induction more flexible for more variable relations is by not simply using the N maximal value as step rather some f (x, y, z, . . .) simplicity function.
So we prove as induction step that all u, v, w, . . . being in R and also obeying:
f (u, v, w, . . .) < f (x, y, z, . . .) implies the inheritance of R to x, y, z, . . .
Or as infinite descent we show that an x, y, z, . . . counter example implies a u, v, w, . . . with smaller f value that is a simpler counter example.
In the usual negative infinite descent we use not R as R. So the R must inherit to simpler cases and we conclude that not R is true for all numbers that is R is impossible.
For Fermat's Last Theorem R is $x^n + y^n = z^n$ and it is still just a three variable relation because unfortunately nobody could use it as four variable, including n. So, we merely use particular n values and then for these individual R, it's still very hard and always different to show an inheritance. The f (x,y,z) simplicity function is usually the same though as x y z.

Euler's idea for R as $x^3 + y^3 = z^3$, that is to find u, v, w so that $x^3 + y^3 = z^3$ implies $u v w < x y z$ and $u^3 + v^3 = w^3$, went as follows:

First simplify x, y, z so assume that these are simple or relative primes.
As we explained, this means also exactly one of them being even.
If it is z then let $x < y$ and let $p = \dfrac{x+y}{2}$ and $q = \dfrac{y-x}{2}$

If one of x , y is the even, then let it be x and let $p = \dfrac{z-y}{2}$ and $q = \dfrac{y+z}{2}$

Calculating $2p(p^2 + 3q^2)$ we get z^3 in the first case, x^3 in the second.

Showing just the first case: $2p(p^2 + 3q^2) =$

$$(x+y)\left(\frac{x^2+y^2+2xy}{4} + 3\,\frac{y^2+x^2-2xy}{4}\right) = (x+y)(x^2+y^2-xy) =$$

$$(x+y)\left((x+y)^2 - 3xy\right) = \left((x+y)^3 - 3x^2y - 3xy^2\right) = x^3 + y^3 = z^3$$

So $2p(p^2 + 3q^2) =$ cube no matter what. Also observe that in both cases:
1.) $p + q = y$ is odd, so p , q have opposite parity and so $p^2 + 3q^2$ is odd too.
2.) p and q are relative primes because their sum y, and difference x, are too.

We claim that $2p$ and p^2+3q^2 can only have c common prime factor as 3.
Obviously c can't be 2 because p^2+3q^2 is odd. So c must divide p.
Thus, dividing p^2+3q^2 it must divide $3q^2$ too. But it can't divide q because p and q are relative primes, so it must divide 3 that is be 3.
So, we have two cases. When $2p$ and p^2+3q^2 are relative primes and when they are both 3 multiples. This second case easy from the first so we continue with them being relative primes.
This means that the they are cubes themselves: $p^2+3q^2 = r^3$ and $2p = s^3$
Now comes the crucial point to calculate this r cubic root.
Using $\sqrt{-3}$ as imaginary square root, it's easy to split p^2+3q^2 into two factors at least:

$$p^2+3q^2 = (p+q\sqrt{-3})(p-q\sqrt{-3}) = r^3$$

Assuming that both factors are cubes and have same form of cubic roots, then:

$$p+q\sqrt{-3} = (a+b\sqrt{-3})^3 = a^3+3a(b\sqrt{-3})^2+3a^2b\sqrt{-3}+(b\sqrt{-3})^3 =$$

$$= (a^3-9ab^2) + (3a^2b-3b^3)\sqrt{-3}$$

Thus assuming also that these imaginary compositions are unique,

$p = a^3-9ab^2$ and $q = 3a^2b-3b^3$. Then

$$2p = 2(a^3-9ab^2) = (a^2-9b^2)2a = (a+3b)(a-3b)2a = s^3 \quad \text{or}$$

$$2p = 2(a^3-9ab^2) = (9b^2-(-a)^2)(-2a) = (3b-a)(3b+a)(-2a) = s^3.$$

If a, b are non zero wholes then in one of these forms we have all positive factors.
These factors are all relative primes so they are cubes.
Also, two of the factors add up to the third and so they can be used as the u, v, w descending cases. It's also easy to see that $uvw < xyz$.

A very sleek proof but with a lot of assumptions in the middle.
Going to these new complex integers of the form $x+y\sqrt{-3}$ seems a simple analogue of the ones we saw as connectors of the plane's infinite tiling. But of course that picture is meaningless for these. But that's not the real problem. We could follow the rules as analogues.
Remember that among those plane numbers we had four units $1, -1, \sqrt{-1}, -\sqrt{-1}$.
They are the 1 length connectors corresponding to right, left, up, or down.
We also saw that primes appear among the connectors too.
The horizontal or vertical 3, 7, 11, . . . $4k-1$ prime number length connectors remain primes as connectors too, but 2, 5, 13, . . . loose their primality. What we didn't continue to show is that among the connector primes the Unique Prime Factorization also stands.
A special easy splitting of a normal number into plane components was the symmetrical pairs.
And indeed Euler used the same above as $p^2+3q^2 = (p+q\sqrt{-3})(p-q\sqrt{-3})$.
In the plane that's the only way, symmetrically, to find factors outside, due to the Unique Prime Factorization. If it were not unique then using only the symmetrical half is an incorrect step.
The next step that is assuming a cubic root for the half $p+q\sqrt{-3}$ is justifiable by assuming that the two halves are relative primes. And finally calculating the cube from an $a+b\sqrt{-3}$ and equating the two components is also a fair uniqueness that doesn't depend on the U.P.F.
So, most of the steps were okay but still the crucial split was not.

Now I show that this can not be fixed because Unique Prime Factorization fails among the $p + q\sqrt{-3}$ combinations. Among the connectors 2, the smallest prime, became a composite as $2 = (1+\sqrt{-1})(1-\sqrt{-1})$. Here 2 remains a prime. So $4 = 2 \bullet 2$ is a prime factorization of 4. But here is a different : $4 = (1 + \sqrt{-3})(1 - \sqrt{-3})$.

Aside from these abstract considerations, the only important fact among the naturals that we need to fix up Euler's proof is that:
$p^2 + 3q^2 = r^3$ implies the existence of a, b non zero wholes that $p = a^3 - 9ab^2$.
And this fact is true. There are always such a, b numbers.
The start of a different approach is the observation that $r = a^2 + 3b^2$ is true too.
This follows from the faulty symmetrical halving too because the other half of the assumed

$p + q\sqrt{-3} = (a + b\sqrt{-3})^3$ is $p - q\sqrt{-3} = (a - b\sqrt{-3})^3$ So multiplying them:

$$p^2 + 3q^2 = (a^2 + 3b^2)^3 = r^3$$

Having a bit of an algebraic experience with cubings, we might even jump up and say eureka, that's all we need because $r = a^2 + 3b^2$ implies $p = a^3 - 9ab^2$, $q = 3a^2b - 3b^3$:

$$r^3 = (a^2 + 3b^2)^3 = a^6 + 9a^4b^2 + 27a^2b^4 + 27b^6$$

$$r^3 = p^2 + 3q^2 = (a^3 - 9ab^2)^2 + 3(3a^2b - 3b^3)^2 =$$

$$a^6 + 81a^2b^4 - 18a^4b^2 + 27a^4b^2 + 27b^6 - 54a^2b^4$$

The two forms of r^3 became the same but this doesn't really prove what we claim.
Indeed, only the combined total became the same, so we could imagine different p and q expressions using a, b so that $p^2 + 3q^2$ is still the same. Even worse is that a fix r^3 could be imagined as allowing different a, b values in the first line, that is $r = a^2 + 3b^2$ is not unique either. So then the obtainability of p, q in the same way for different a, b is even more unlikely. And yet exactly this is true.
Since $r^3 = p^2 + 3q^2$ and also $r = a^2 + 3b^2$ thus these square plus triple squares are a fix form just like the square sums were. So then just as those were the old original world before the plane numbers as connectors, here too these forms must have a world without the imperfect and not even visible complex forms.
The square sum world was very well investigated without the Gaussian Integers that is the connectors, and most amazingly those facts and arguments are perfect template to describe these $a^2 + 3b^2$ forms. In fact we could regard quite generally $a^2 + mb^2$ forms, but we just stick to the $m = 3$ case. We told a lot about square sums but now we repeat their facts too to see the perfect similarity with these that we call the "formable" numbers. Already this expression is an improvement toward clarity as we will see soon. The square sums also should be called square summable numbers. A more substantial change is that as a, b we regard not merely natural but integers, that is negative wholes and 0 too. The allowing of 0 was already a custom among the square summables thus allowing the squares to be included. I always avoided this in my books.
The real goal is to see the cases of naturals even here at formables, so why is the widening okay here, you may ask. This boils down to real and phony abstractions, the most crucial point of didactical correctness. There is only one point where the square summables must be allowed to contain the squares, so the allowing of 0 as a, b is phony among these.
Here among formables the role of not only 0 but even the negatives is essential. In fact, it is the whole essence of how Euler's mistake can be fixed. This allowing of negatives seems even

pointless first. Indeed both a and b are squared in $a^2 + 3b^2$, so any negative has positive version with same full value. When we'll come to the crucial point it will make perfect sense.
The allowing of 0 of course is quite different, it make sense right from the start. So we first only include this and when we come to the right point we'll allow negative a , b too.
All this is not that big deal because I will also use three crucial adverbs to specify the square summables and formables. And the first "strictly" means using only naturals.
The second is "simply" meaning that a and b are relative primes.
The third is "uniquely" meaning that there is only one form for that number. But this is meant in the way that the a^2 and b^2 are unique. Indeed the positive negative variants of a and b can not be excluded so they are regarded as unique. The first two specializations are adjectives for square sums or forms themselves as "strict" and "simple". Unique of course is trivial for a single form so we won't use it. To see better all this, lets see the beginning of both numbers:

Square summables:

$0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$, $4 = 0^2 + 2^2$, $5 = 1^2 + 2^2$, $8 = 2^2 + 2^2$, $9 = 0^2 + 3^2$

$10 = 1^2 + 3^2$, $13 = 2^2 + 3^2$, $16 = 0^2 + 4^2$, $18 = 3^2 + 3^2$, $20 = 2^2 + 4^2$, $25 = 0^2 + 5^2 = 3^2 + 4^2$

Formables:

$0 = 0^2 + 3 \bullet 0^2$, $1 = 1^2 + 3 \bullet 0^2$, $3 = 0^2 + 3 \bullet 1^2$, $4 = 1^2 + 3 \bullet 1^2 = 2^2 + 3 \bullet 0^2$

$7 = 2^2 + 3 \bullet 1^2$, $9 = 3^2 + 3 \bullet 0^2$, $12 = 3^2 + 3 \bullet 1^2$, $13 = 1^2 + 3 \bullet 2^2$

$16 = 2^2 + 3 \bullet 2^2 = 4^2 + 3 \bullet 0^2$, $19 = 4^2 + 3 \bullet 1^2$, $21 = 3^2 + 3 \bullet 2^2$

In the square summables I only went up to 25 because this is the first non uniquely summable. It is simply square summable, in fact both forms $0^2 + 5^2$ and $3^2 + 4^2$ seem simple but the first is not strict. With an abstract convention that everything divides 0 it is not even simple. This convention makes sense here too, because 5 dividing 0 simply means that we can simplify, that is bring out 5^2 as $5^2(0^2 + 1^2)$.
The double $50 = 1^2 + 7^2 = 5^2 + 5^2$ so here both forms are strict but the second is not simple.
We can have two forms that both are simple.
The first number that has such is $65 = 1^2 + 8^2 = 4^2 + 7^2$.
Among the formables the non unique ones start much sooner at 4.
The second form is not strict and the same happened at 16.
The corresponding case of 25 above, is here $49 = 1^2 + 3 \bullet 4^2 = 7^2 + 3 \bullet 0^2$.
The second is not strict and thus by the convention neither simple.
The first number that is not uniquely formable with two simple forms is:
$91 = 4^2 + 3 \bullet 5^2 = 8^2 + 3 \bullet 3^2$
Lets see the primes.
The common main feature is that they are always strict, simple and unique.
Except 3, the minimal prime of the formables is not strict, it needs 0.
This is the first absolute sign of the necessity to go out of the naturals.
Among square sums, 2 is the minimal prime and the rest are all the $4k + 1$ ones.
Among formables similarly all others than the minimal 3, are all the $6k + 1$ ones.
These amazingly perfect and identical prime rules do not come out easily.
Instead, we have to go into a jungle and bring them back from there.
This fact is crucial because it reveals that this could be the case for Fermat's Last Theorem too.
So this universal simple fact would come out from discovering an even bigger jungle first.
The present proof of Fermat's Last Theorem is not coming back from such jungle.
This jungle must be big but still crystal clear and simple common sense through naturals.

We can extend the naturals step by step as we do too for this much simpler task of proving why the primes obey such simple rules. But we mustn't extend concepts as crazy just for the sake of being more abstract. Otherwise the derived results among the simple naturals will always leave a hole in our plausibilities. We feel being cheated and we have been cheated in a quite common sense too. Formalism is not merely a didactically incorrect mistake. It is an evil human act.
Some formalisms do provide "silver platters", easy and instantly visible realities.
Best example is our number system.
At its first step, the use of zero simply makes the naturals a much better calculable reality.
The roman soldiers were unable to calculate the taxes in roman numerals. And yet an early Pope called the zero the devil's work. Practicality of course prevailed. But the real advantage came later with the decimal point. The infinite decimals are showing how finite lengths are combined from infinite many smaller and smaller lengths. Achilles paradox was an earlier verbal contradiction provoked from this fact by using the infinite many members in time. Indeed, if infinite many times the sum is still not the total then we could say that it is never reaching it. This of course is false. Infinite many times doesn't mean for ever. In fact, there are infinite many times before any "now". Indeed, a minute ago, half minute ago, a quarter of a minute ago, and so on, are approaching the "now". Behind this lies the bigger assumption that time is just like space.
Leaving this problem now, we can instead focus on the third step in our silver platter. Namely that the division algorithm for naturals, that is calculating the fractions is always periodic because there can be only finite many possible remainders. This instantly shows that the non periodic infinite decimals are not fractions. An other old Greek problem cleared instantly.
The wider analogue of these mathematical silver platters is technology for physics.
And indeed, technology did make certain realities of physics directly plausible.
But matter and technology is a common bigger loop.
The true connection is direct and instant, both in math and physics. Namely, through the act of understanding. Understanding the universe is the goal, not using it to replace understanding.
Doing math or science without the intention of making understandings of it, brings our whole activity down to mere adaptation, actions to achieve advantage. No matter how fancy and noble our feelings are about the role of science, without striving to make others understand, all our actions are exactly as of rats in the lab. And someone is watching us too!
To show the first split of primes, the $4k+1$ ones as square sums, we could stay among the naturals that is regard only strict square sums. For the second split, the $6k+1$ ones as formables we have to accept at least 0. But then to fill up Euler's gap we have to go to integers. So regarding integers from the start is fair. In fact it will show new truths about the square sums too.
This shows that we are always doomed in a sense didactically if our goal is to merely prove individual things. And so you might say that the crazy over abstractions are merely a potential bigger view to prove more things later. The reason this is not so is that understanding, seeing, is always concrete. So the problem with crazy textbook writers is not that they are insanely abstract but that they see things that they don't tell. Namely, exactly how those abstractions make concrete sense. So they are not insane, they are common liars, immoral scumbags.
I try not to be one. And here, trying at once leads to perfect absolution. It is a single moment of clarity to become aware of the potential lying in our communications in general. Trying to explain better then becomes a course of actions that was completely suppressed before.
It is truly an awakening. But it brings about seeing the bigger darkness of our present. So if you want to be happy than better remain a scumbag.
The reward of being honest is not merely to feel good. The avoidance of crazy unexplained abstractions to others, brings about new abstractions in ourselves. This makes perfect sense. Filtering our abstractions through communicability makes us see even more of them. This brings about a new conflict though. How do we explain those. You'll find the answer to this yourself.
The basic law of square summables and formables or even their generalizations is that products of such is always the same.
Recall the equality that proves this and the rule for calculating the product from page 5:

$$(a^2 + b^2)(c^2 + d^2) = (a\,c + b\,d)^2 + (a\,d - b\,c)^2.$$

Observe the rule:

$$\begin{tikzcd} a \arrow[d] & b \arrow[d] \\ c & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[dr] & b \arrow[dl] \\ c & d \end{tikzcd}$$

$$+ \qquad\qquad\qquad -$$

The general situation is similar:

$$(a^2 + m b^2)(c^2 + m d^2) = (a\,c + m\,b\,d)^2 + m\,(a\,d - b\,c)^2.$$

The rule is:

$$\begin{tikzcd} a \arrow[d] & b \arrow[d] \\ c & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[dr] & b \arrow[dl] \\ c & d \end{tikzcd}$$

$$+\,m \qquad\qquad\qquad -$$

This is our situation with $m = 3$.
Now the more logical question is whether the reverse is true too. But in what sense?
All formables merely being products of such is obviously impossible. We have to accept new original ones that have no such factors simply because prime numbers have no factors at all.
So the square summable or formable primes are obviously such original, God given concrete facts. And indeed, there is no formula that would give the two members for these primes.
This makes it even more amazing that simple formulas $4k + 1$ and $6k + 1$ rule their sets.
A weaker reversal of the product rule could be that at least when a square summable or formable has some factors then they are same too. Of course this is not true either.
Among square summables the first counter example is 9, among formables it is 4.
Indeed, 3 is not square summable and 2 is not formable.
So the question is this: Why are these new composites entering and what are the ones that only have perfect factors, that is being also square summable or formable?
We might blame everything on the 0, because it allows all square numbers in both sets of numbers and a square can have anything as factor. But this is an over simplification. We will find ugly strict ones too. In fact $18 = 3^2 + 3^2$ and $4 = 1^2 + 3 \bullet 1^2$ are easy examples.
The first is not simple but the second is and 4 has no formabe factor. All such counter examples among the formables are even. So the "simple" rule is:
Simply square summables and odd simply formables are the perfect ones.
They inherit their square summability and formability to all their factors if there are factors.
Looking more closely at these factors we can realize that they are not only always square summables and formables but also again simply. So amazingly, the simplicity or rather the possible simplicity inherits to the factors too. This might suggest that we were going in wrong directions from the start and should have only regarded simple square sums and forms. After all these are the closed world in themselves, products and factors remain the same. But we made a mistake here. We never proved that products of simples remain simple. And as we'll see it's not true. So we do have to look among all forms. The true description of when the simple ones multiplied remain simple comes from the primes. Namely the role of the minimal primes, 2 among the square summables and 3 among the formables. We might jump to the conclusion that it has to do with the mentioned coincidence rules for the non minimal primes as $4k + 1$ and $6k + 1$. But we are wrong, these amazing rules have no roles in explaining the simplicity loss of products. So, we don't have to know that the remaining non minimal square sum primes are all the $4k + 1$ and formables are the $6k + 1$ ones. Without knowing these, it can be shown that using the non minimal primes the simplicity remains, even multiplying more of the same primes and thus obviously having square factors. So:
Any products made from 5, 13, 17, 29, . . . will be simply square summable.
Any products made from 7, 13, 19, 31, . . . will be simply formable.

As I listed them, I myself went by the rules I know as $4k+1$ and $6k+1$ but these are not the essential facts to prove this, rather that the firsts were all square sums and the second were formables. The blocking out of 2 and 3 was not an absolute necessity! We can allow these to be used but only once. In short, having 4 or 9 as factor is that ruins simplicity.
For 2, this anomaly is perfectly visible from the connector picture. Indeed, the factorization of 2 as $(1+\sqrt{-1})\,(1-\sqrt{-1})$ is special because these two symmetrical primes are amazingly the "same", namely merely unit variants of each other: $1-\sqrt{-1} = -\sqrt{-1}\,(1+\sqrt{-1})$.
For no other primes do we have this because they form pairs with other than 45 degrees.
Thus, to split any products made from 5, 13, 17, 29, . . . into two symmetrical groups in the plane is possible without having symmetrical members and thus simplifying naturals as products in either group. But the two factors of 2 can only be split once. A second 2 factor would mean necessarily symmetrical picks causing 2 appear in both groups and thus having only square sums simplifiable by 4. We can not see this for 3 and formables, but it is true here too.
Knowing the simple ones among our numbers we at once know the non simple ones, that is the ones that are formable but never simply. These are simple ones multiplied by a square that contains the minimal or outside prime as factor. Indeed, it obviously remains formable because a square can be taken into the members and can not be simple because it will contain non formable or non simple factor. Namely, the outside prime or the minimal prime square.
The proof of the grand claim of inheritance to factors among simples, goes first for prime factors. For the square summables, this can go easily through the connector vision and its unique prime factorization. But there is a very elegant proof with simple induction too and it translates to formables. The connector proof goes like this:
x^2+y^2 being a simple square sum with p prime factor means that p divides $x^2+y^2 = C\overline{C}$ where these are simple that is minimal too. So p can not divide them separately.
This means that p is not prime as connector. Indeed among connectors the same rule applies that primes divide separately. So it must be a product. It can only be product of two symmetrically angled factors as any horizontal connector. But since p is prime among the naturals, these two factors can not have natural factors because they would have to divide p.
So the two factors are minimals. But symmetrical angled minimals are same lengths too. So these two factors are actually symmetrical. So, $p = (a+b\sqrt{-1})\,(a-b\sqrt{-1}) = a^2+b^2$.
Once we know that the prime factors are all formables, then their products can be proved to be inheriting the simple formability either step by step for using the non minimal primes only, or for square sums at once with the argument I showed above with the group separation.
The $4k+1$ and $6k+1$ coincidences are proved only through didactically questionable tricks.
We create simple square sums or forms that have the $p = 4k+1$ or $p = 6k+1$ prime as factor.
There are more theorems that allow such concrete square sums or forms for p.
A more systematic argument regards the remainders to p primes and then shows pairs.
For example, for $p = 4k+1$ if a q remainder is square, that is $q = [r^2]$ then $p-q$ is too.
But $q = 1$ is an obvious square and so $p-1$ is such too. Thus $p-1 = [r^2]$ which means that $r^2 = mp + p - 1$ and so p divides r^2+1 which is a special simple square sum.
Pretty ad hoc proofs for something that beautiful.
But now comes a third line that wasn't even used among square sums only developed for the forms to deal with Euler's gap. This line is pursuing uniqueness among the simples. Real uniqueness doesn't stand but a relative one does. This is where the step from naturals to include negative numbers is an absolute must.
Lets recall the first non unique square summable and formable numbers with two simple forms.
$65 = 1^2+8^2 = 4^2+7^2$ and $91 = 4^2+3\bullet 5^2 = 8^2+3\bullet 3^2$.
We really cant see any connections between the two forms in either case.
And yet the connection is that they both come out with our established product manufacturing rules if we allow negative input numbers too. Of course we need the numbers themselves looked as products of their prime factors which have their own empirical forms too:
$65 = 5\bullet 13 = (1^2+2^2)\,(2^2+3^2) = 1^2+8^2 = 4^2+7^2$
$91 = 7\bullet 13 = (2^2+3\bullet 1^2)\,(1^2+3\bullet 2^2) = 4^2+3\bullet 5^2 = 8^2+3\bullet 3^2$

This didn't help much and a much better thing is to envision the forms of the factors under each other because our rules were formulated this way:

$$1^2 + 2^2 \qquad\qquad \text{and} \qquad\qquad 2^2 + 3 \bullet 1^2$$
$$2^2 + 3^2 \qquad\qquad\qquad\qquad\qquad 1^2 + 3 \bullet 2^2$$

And indeed $1 \bullet 2 + 2 \bullet 3 = 8$ and $1 \bullet 3 - 2 \bullet 2 = -1$

So the negativity entered but in a trivial sense because the squares are 8^2 and 1^2.

But at least we obtained the first product form. A bit of trial and error will soon show that using negatives for the input numbers as well, can give the second form too namely as:

$$(-1) \bullet 2 + 2 \bullet 3 = 4 \quad \text{and} \quad (-1) \bullet 3 - 2 \bullet 2 = -7.$$

To avoid lucky trials is easy. We should list all possible combinations for the two factors:

1	2	−1	2	1	−2	−1	−2
2	3	2	3	2	3	2	3
1	2	−1	2	1	−2	−1	−2
−2	3	−2	3	−2	3	−2	3
1	2	−1	2	1	−2	−1	−2
2	−3	2	−3	2	−3	2	−3
1	2	−1	2	1	−2	−1	−2
−2	−3	−2	−3	−2	−3	−2	−3

Using our formulas we always get only the $1^2 + 8^2$ or $4^2 + 7^2$ resulting forms.

The same will stand for the formables.

Using more prime factors we have more and more different forms and they all come out from the even more possible sign combinations.

This also means that to get the possible forms of an $M\,N$ product made from formables, we merely have to make combinations of the M and N forms.

This then also means that in reverse too, knowing the product's one form and one form of one of the factors, the other will have forms that give the result this way.

Finally for powers this means that any form of a power, there are sign variation forms of the base, so that they give the power by our rules.

So $r^3 = p^2 + 3\,q^2$ not only means that r as factor of a simply formable will be also simply formable, but that r will have an $a^2 + 3\,b^2$ simple form, so that three sign variations of this must give as product $p^2 + 3\,q^2$ using the rules. Then of course all sign variations give the same except with negative variants for p or q.

The usual Euler forms of p and q come out with the following sign choices:

$$p^2 + 3\,q^2 = (a^2 + 3\,b^2)\,(a^2 + 3\,(-b)^2)\,(a^2 + 3\,b^2).$$

Indeed, the first two multiplied by our rules gives:

$$(a^2 + 3\,b(-b))^2 + 3\,(a\,(-b) - b\,a)^2 = (a^2 - 3\,b^2)^2 + 3\,(-2\,a\,b)^2.$$

This then multiplied with the last third factor by the rules gives:

$$[\,(a^2 - 3\,b^2)\,a + 3\,(-2\,a\,b)\,b\,]^2 + 3\,[\,(a^2 - 3\,b^2)\,b - (-2\,a\,b)\,a\,]^2 =$$

$$a^3 - 9\,a\,b^2 \qquad\qquad\qquad\qquad 3\,a^2 b - 3\,b^3$$

$$p \qquad\qquad\qquad\qquad\qquad\qquad q$$