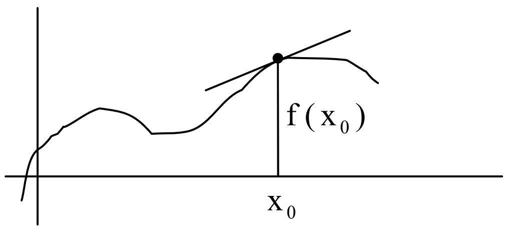
Cauchy Riemann Equations

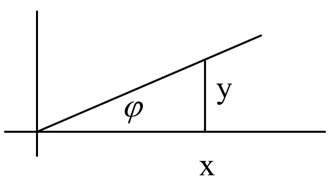
R There is a simplest meaning of these equations, that is always missed behind the more complicated, higher arguments.
A wider misrepresentation of the differential ratio should be cleared first.

D If X is an abstract variable giving Y values through some F function, that is as: Y = F (X), then a fundamental question is how F can be approximated by a simpler function. Sticking only around an X_0 value, quite generally a so-called homogene linear approximation of F is the $F'(X_0)$ function. The derivative of F at X_0.
The word "linear" refers to lines and indeed means that lines as point sets are taking up values that form again lines, and also parallel lines become parallel. In short, these two mean that angles are changed into same angles.
The word "homogene" means that the origin of the coordinate system is ordered to an origin too. This third feature at once means that $F'(X_0)$ is not an approximation of F, rather only regarding F from the $(X_0 , F(X_0))$ new origin. In other words, regarding the $F(X) - F(X_0)$ change as function of the $X - X_0$ change:
$F(X) - F(X_0) \cong F'(X_0)(X - X_0)$ or in short $\Delta F \cong F'(X_0)(\Delta X)$.
From this homogene "change" approximation we can at once obtain the non homogene real approximation of F (X) by simply taking $F(X_0)$ to the other side:
$F(X) \cong F(X_0) + F'(X_0)(X - X_0)$
All these ideas imply that among the X and Y variable values, the lines, parallelity, addition and subtraction are somehow meaningful.
The simplest scenario is if both X and Y are x , y number values.
The Descartes system allows the viewing of y = f (x) functions in the plane:

The approximation at an x_0 place is a touching or tangent line:

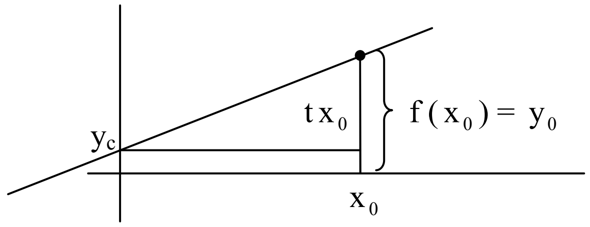


$f(x) - f(x_0) = \Delta f \cong f'(x_0)(\Delta x) = f'(x_0)(x - x_0)$
Since the $(x_0 , f(x_0))$ point is regarded as new origin, thus the touching line can be simply determined by its angle or slope. This slope is strangely, again called as "tangent" of the actual angle and is simply the y / x ratio on the line:

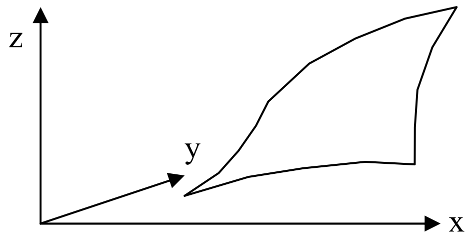


$\tan \varphi = \dfrac{y}{x}$

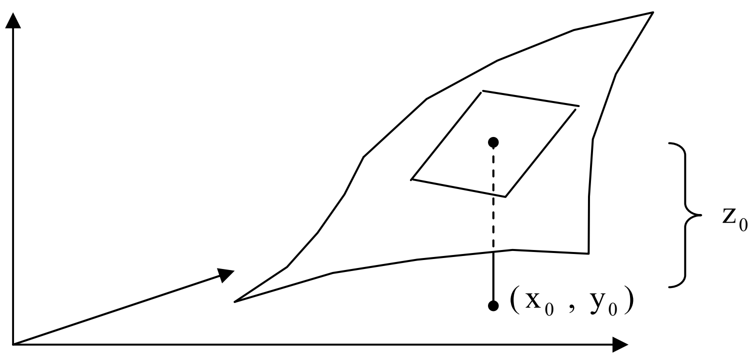
Then, the $f'(x_0)$ linear approximation is simply a multiplication by the t tangent value: $f(x) - f(x_0) = \Delta f = f'(x_0)(\Delta x) = t \bullet \Delta x = t(x - x_0)$
The full, approximating local line is:
$f(x) = f(x_0) + t(x - x_0) = tx + f(x_0) - tx_0 = tx + y_0 - tx_0 = tx - y_c$.



So it's quite visual how this is the same t sloped line but crossing the y coordinate line at $y_c = y_0 - tx_0$.
As a "simplification" in notation, the t slope or tangent multiplier became directly regarded as the $f'(x_0)$ approximating function. So, the multiplication as process became replaced by the multiplier. In fact, usually it is expressed on its own as:
$f'(x_0) = t \cong \dfrac{f(x) - f(x_0)}{x - x_0}$ or defined exactly as $\lim \dfrac{f(x) - f(x_0)}{x - x_0}$.
Since $f'(x_0)$ is merely a multiplication, thus, its real meaning as an approximative homogene linear function, had been lost in lower education, namely in high school.
The linearity is indeed trivial, because the x axis is itself a single line. So only the homogenity is that still remains and allowed the single multiplier. This burying of the basic idea then has to be dug out in tertiary level education. This is a joke, because the whole meaning of the approximative local function would be to explain Newtonian mechanics. But as we all know, physics' education today is totally unrelated to math's. This is the irrefutable total failure of science education as such.
The simplest next scenario after the $y = f(x)$ number functions is $z = f(x, y)$.
This still can be viewed in the three dimensional Descartes space as a surface hovering over the x, y coordinate plane and giving as heights the z values:

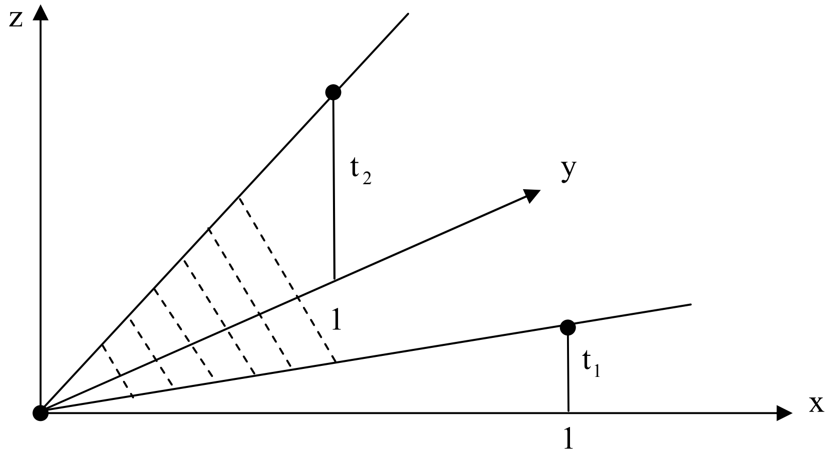


The approximation now means a touching plane at $(x_0, y_0, z_0 = f(x_0, y_0))$



This can be reduced to the number-number approximations, through a simple geometrical fact:

T A plane through the origin and having t_1 , t_2 slopes to the x , y axis, has the equation $t_1 x + t_2 y = z$.



P $(0, 0, 0)$, $(1, 0, t_1)$ and $(0, 1, t_2)$ satisfy the equation.
These three determine exactly the mentioned plane.

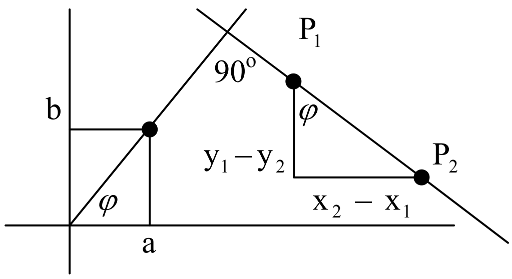
T The touching plane of $z = f(x, y)$ at $(x_0, y_0, z_0 = f(x_0, y_0))$ is:
$t_1 x + t_2 y = z$ with (x_0, y_0, z_0) as origin and
$t_1 = [f(x, y_0)]'(x_0) = f'(x_0)$, $t_2 = [f(x_0, y)]'(y_0) = f'(y_0)$.

P The t_1 value is the slope of $f(x, y_0)$ at x_0 because we keep y_0 fix.
Similarly for t_2, we keep x_0 fix.

R Thus, the $\Delta f(x, y)$ change is now not merely a multiplication, rather a combination of two: $\Delta f(x, y) = t_1 \Delta x + t_2 \Delta y = f'(x_0)\Delta x + f'(y_0)\Delta y$.
So linearity shows its simplest form. Indeed, now the two features can be seen:
Lines of the x , y plane remain lines, when projected up to the touching plane.
And also, parallel ones remain parallel.
The next case is ordering to (x, y) points of a plane two values.
So, the three dimension is not enough to view this, but this is not a disadvantage, rather a step back to reality:
The ordered values must be regarded as coordinates in the same plane as (x, y).
So we have an $F = (x, y) \rightarrow (x', y')$ transformation of the plane.
In fact, F means two number valued functions: $x' = f(x, y)$, $y' = g(x, y)$.
(These apostrophes are not to be confused with the notation of the derivatives, where it always follows a function: F' , f' , g' and so on.)
This F will be now a two dimensional homogene linear transformation of the plane.
But from our previous results, we simply have:
$\Delta x' = f'(x_0)\, \Delta x + f'(y_0)\, \Delta y$ and $\Delta y' = g'(x_0)\, \Delta x + g'(y_0)\, \Delta y$
Omitting Δ-s and using α , β , γ , δ for $f'(x_0), g'(x_0), f'(y_0), g'(y_0)$:
$x' = \alpha x + \gamma y$ and $y' = \beta x + \delta y$
Such simple first order combinations of x and y are also called homogene linear.
But, we had already a geometrical definition of linearity as keeping lines and parallels, so we have to show that this new algebraic definition means the same.
Without homogenity we have additional fix ω_1 and ω_2 members:

T An $(x, y) \rightarrow (x', y')$ transformation is linear, if and only if:
$x' = \alpha x + \gamma y + \omega_1$
$y' = \beta x + \delta y + \omega_2$

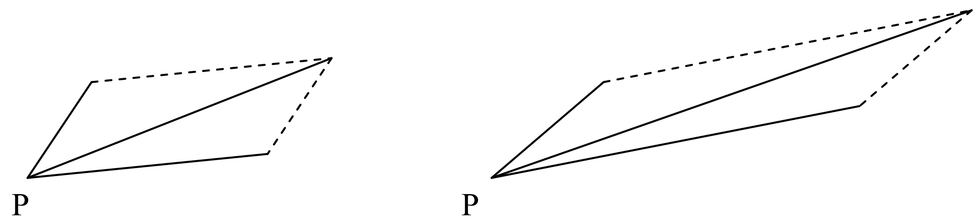
P First we'll show that this is linear in the geometrical sense.
For this we need to see what lines are algebraically.
Claim:
For any two $P_1 (x_1 , y_1)$, $P_2 (x_2 , y_2)$ points that satisfy an $a x + b y = c$ equation, the $P_1 P_2$ line is perpendicular to the (a , b) point connected to the origin.



$a x_1 + b y_1 = c$, $a x_2 + b y_2 = c$ $\rightarrow$ $a(x_2 - x_1) + b(y_2 - y_1) = 0$. So:

$\frac{a}{b} = -\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_2 - x_1}$ which means exactly that the two φ are same.

This implies that $a x + b y = c$ is an equation of a line, perpendicular to (a , b).
Two such $a x + b y = c$ and $A x + B y = C$ lines are parallel iff $A = \lambda a$, $B = \lambda b$.
Not only these are lines, but all lines are such. Indeed, we can always find (a , b) perpendicular direction and picking an (x_0 , y_0) point on the line, putting these into the equation, we get the $a x_0 + b y_0 = c$ value.
Now it's easy to see that algebraically linear transformation of an $a x + b y = c$ line is again a line and parallel ones transform into parallel too. We simply have to replace x , y by x' , y' expressed with $\alpha , \beta , \gamma , \delta , \omega_1$ and ω_2.
The reverse is to show that all geometrically linear transformations must be expressible with $\alpha , \beta , \gamma , \delta , \omega_1$ and ω_2.
The conservation of lines and parallelity implies that the sum of vectors becomes the sum of the transformed vectors:



Applying this to $P = O$ and $P' = O' = \Omega = (\omega_1 , \omega_2)$:
All points of the plane then can be combined from the coordinates as rectangles that transform into parallelograms from Ω.
This implies that addition of the units on the coordinates remain additions and then any fractions and rational proportions remain the same too. Then by continuity all coordinate values transform proportionally. In short, the two coordinates become slanted and stretched from Ω. Most importantly though the two coordinates can have

different stretches. This implies that all lines from O will be stretched from Ω and their stretch is also determined from the coordinate stretches.
The calculation of a transformed point is easy as:

$P' = (x, y)' = x(1, 0)' + y(0, 1)' + O' = x(\alpha, \beta) + y(\gamma, \delta) + \Omega =$
$(\alpha x, \beta x) + (\gamma y, \delta y) + (\omega_1, \omega_2) = (\alpha x + \gamma y + \omega_1, \beta x + \delta y + \omega_2) = (x', y')$

This proved our claim. In the special case, when the coordinates are stretched the same and also turned with the same angle, then all points must be stretched and turned this same way. But to see this from α, β, γ, δ is not obvious.
Regarding only homogene linear transformations:

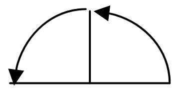
$$\left.\begin{aligned} x' &= \alpha x + \gamma y \\ y' &= \beta x + \delta y \end{aligned}\right\} \text{ is abbreviated as } \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \text{ matrix}$$

This helps to recognize turnings. For example, a very special one as start:

T The $x' = -y = 0x + (-1)y$
$y' = x = 1x + 0y$ transformation,

that is, the $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ matrix turns with 90^o.

P $(1, 0) \rightarrow (0, 1) \rightarrow (-1, 0)$

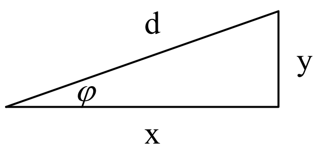


T $\begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix}$ is combining the transformed point from two components.
One merely stretched by λ and one turned 90^o and stretched by θ.
In short, $P' = \lambda P + \theta P_{90}$.

$(1, 0) \rightarrow (\lambda, \theta) = \lambda(1, 0) + \theta(0, 1) = \lambda(1, 0) + \theta(1, 0)_{90}$
$(0, 1) \rightarrow (-\theta, \lambda) = \lambda(0, 1) + \theta(-1, 0) = \lambda(0, 1) + \theta(0, 1)_{90}$

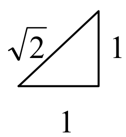
So it does what we claim for the $(1, 0)$, $(0, 1)$ units and thus, it does the same for all linear combinations of these.

R For an amazing alternative meaning of this same matrix, we have to introduce the two basic trigonometric functions. The earlier mentioned $\tan\varphi$ is the third.
This we called the slope, but we should call it the theoretical slope. It gives the proportion of the elevation, relative to the forward motion. If we are driving on an ascending road, then the road sign showing twenty percent slope, would mean twenty percent elevation, not relative to the forward distance, which is hidden under the road, rather to the actual distance we travel. This slope is called the sine = $\sin\varphi$:

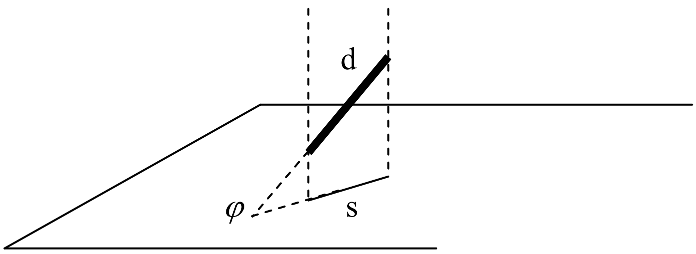


$\tan\varphi = \frac{y}{x}$ = theoretical slope , $\sin\varphi = \frac{y}{d}$ = practical slope

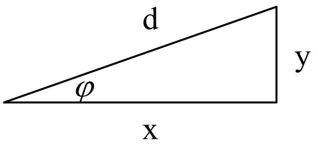
For small angles, the two measures of slopes are very close, but already at $\varphi = 45^o$ they are quite different. Indeed, $\tan 45 = 1$ while $\sin 45 = \frac{1}{\sqrt{2}}$:



At higher angles, the difference becomes drastic, because as we approach 90^o, the tangent becomes ∞, but the sine, only 1.
The second basic trigonometric function, cosine, is not measuring a slope, rather the s shadow of a d distance that has φ angle to the horizontal plane.
We assume that the sun is exactly above, like at noon:



Or, in a simpler two dimensional sense, $\cos\varphi = \frac{x}{d}$:

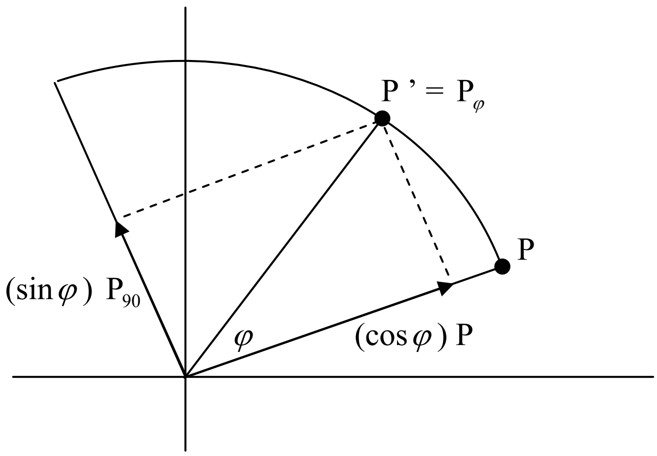


The Pythagoras Theorem gives a relation between sine and cosine:

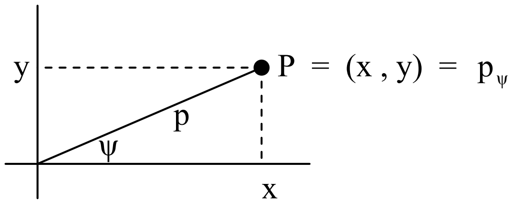
$$(\sin\varphi)^2 + (\cos\varphi)^2 = (\frac{y}{d})^2 + (\frac{x}{d})^2 = \frac{x^2+y^2}{d^2} = \frac{x^2+y^2}{x^2+y^2} = 1$$

T $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is a turning with φ. That is, P ’ = P_φ

P By our previous theorem, P ’ = $(\cos\varphi)$ P + $(\sin\varphi)$ P_{90} and this is indeed P_φ.



R Just as the one dimensional, that is number number functions’ derivative became oversimplified due to the linear approximation becoming a multiplication and then this regarded as a mere number, the same happened among the plane transformations. Of course, for this to make sense, we would have to have a multiplication for points of the plane. Indeed, such multiplication can be defined by regarding the points not in their usual Descartes coordinates, rather in so-called polar representation. This means giving the angle and the distance from the origin:



The product of two P , Q points is then simply the point that has its length as the product of the lengths, but its angle as the sums of P and Q. So:

$P\,Q = p_\psi\, q_\varphi = (p\,q)_{\psi+\varphi}$

This multiplication has amazing features and allowed the plane points to be regarded as so-called complex numbers.

This had applications even just for the "whole" values, that is for the grid points of the plane. Namely, it shed light on some features of the prime naturals. Indeed, the primes can be defined among these grids and they include some, but not all, natural primes.

More importantly, the complex numbers entered physics at every level. From electro-magnetic equations, to quantum mechanics. A newest application are the fractals.

For us, the important thing is that multiplication of a variable P point by a fix Q, is a homogene linear transformation. Then the logical question is what the condition on a general $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ homogene linear transformation is to be a simple multiplication:

$$\left.\begin{aligned} x' &= \alpha x + \gamma y \\ y' &= \beta x + \delta y \end{aligned}\right\} \leftrightarrow P' = (x', y') = PQ = (x , y)\, q_\varphi = q\,(x , y)_\varphi$$

We already established that $(x , y)_\varphi$ is $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, so the answer is easy:

T $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ is a multiplication by q_φ if and only if $\delta = \alpha$ and $\gamma = -\beta$.

With positive α , β , the matrix has to be : $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

P Let $q = \sqrt{\alpha^2 + \beta^2}$ and let $\dfrac{\alpha}{q} = \cos\varphi$ and $\dfrac{\beta}{q} = \sin\varphi$.

Such φ exists because $\dfrac{\alpha}{q}$ and $\dfrac{\beta}{q}$ are not only values between 0 and 1 but also:

$(\dfrac{\alpha}{q})^2 + (\dfrac{\beta}{q})^2 = \dfrac{\alpha^2 + \beta^2}{\alpha^2 + \beta^2} = 1.$ Then in short:

$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} q\cos\varphi & -q\sin\varphi \\ q\sin\varphi & q\cos\varphi \end{pmatrix}$ which indeed transforms as $q\,(x , y)_\varphi$.

T Cauchy Riemann Equations

The $x' = f(x , y)$, $y' = g(x , y)$ transformation's derivative at x_0 , y_0 ,

that is $\Delta x' = f'(x_0)\,\Delta x + f'(y_0)\,\Delta y$, $\Delta y' = g'(x_0)\,\Delta x + g'(y_0)\,\Delta y$

is merely a multiplication iff $f'(x_0) = g'(y_0)$ and $f'(y_0) = -g'(x_0)$