Collatz Conjecture

The crucial steps to exactify Number Theory, were discovered by Peano before exact logic was developed. But now we'll regard the newer details too.
A basic logical axiom accepts that equality of variables implies the same for function values.
$x_1 = y_1 \ , \ x_2 = y_2 \ , \ . \ . \ . \ . \ , \ x_n = y_n \ \rightarrow \ f (x_1 , x_2 , . . . , x_n) = f (y_1 , y_2 , . . . , y_n) .$
A reversal is obviously not true because different combinations can bring about same values.
For example $3 + 4 = 2 + 5$ so $f (x_1 , x_2) = f (y_1 , y_2)$ doesn't imply $x_1 = y_1$ and $x_2 = y_2$.
But even a single variable $f (x)$ can have same values for different x , so $f (x) = f (y)$ does not imply $x = y$. Yet for the counting or 1 addition this is true:
$x + 1 = y + 1 \ \rightarrow \ x = y$. So same numbers must have same previous. This is our first axiom.
As I said above, the reversal $x = y \ \rightarrow \ x + 1 = y + 1$ is a universally accepted logical axiom so we don't need it as special number theoretical.
Our first axiom $x + 1 = y + 1 \ \rightarrow \ x = y$ is always true but it doesn't mean that we always have a previous number. Namely, 1 doesn't have a previous and only 1 doesn't have one.
So this is actually two axioms:
There is no x that $x + 1 = 1$ that is, for every x $x + 1 \neq 1$.
But if $x \neq 1$ then there is a y that $y + 1 = x$.
Using the precise $\forall$ quantor for "every" and $\exists$ for "there is" , our axioms are:
$\forall x \forall y \ (x + 1 = y + 1 \ \rightarrow \ x = y)$
$\forall x (x + 1 \neq 1)$
$\forall x [x \neq 1 \rightarrow \ \exists y (y + 1 = x)]$
We might feel that these three axioms only allow the:
$1 , 1 + 1 = 2 , 2 + 1 = 3 , 3 + 1 = 4 , \ . \ . \ .$ standard natural numbers.
But we are wrong and we can add a new n object.
Of course, the addition as basic operation and 1 as name at once means that we must have:
$n + 1 = n_1 , \ n_1 + 1 = n_2 , \ . \ . \ .$ as new objects too. Plus by our third axiom n must have previous and that again and so on, so we must have a whole double infinite set of new objects:
$N = \ . \ . \ . \ n_{-2} = n_{-3} + 1 \ , \ n_{-1} = n_{-2} + 1 \ , \ n = n_{-1} + 1 \ , \ n_1 = n + 1 \ , n_2 = n_1 + 1 \ . \ . \ .$
In fact, we can have many such alternative non standard N sets added to the standard naturals.

Peano defined addition, multiplication and exponentiation by recursive axioms from which all values can be actually computed.
In elementary school we just accept that addition is repeated counting :
$x + y = x + 1 + 1 + \ . \ . \ . + 1$ with y many 1 additions.
Then $x y = x + x + \ . \ . \ . + x$ with y many x additions.
And $x^y = x \ x \ . \ . \ . \ x$ with y many x multiplications.
These are not exact because we shouldn't use "dots" in a language.
Peano avoided this imprecision even before Logic demanded it.
An amazing later result showed that exponentiation is actually unnecessary. But it also showed that multiplication brings in a whole new level of complexity beyond addition.

Peano's recursive addition axiom for the other than 1 addition values is:

$x + (y + 1) = (x + y) + 1.$ To see its use, lets prove that $4 + 3 = 7$.

$4 + 2 = 4 + (1 + 1) = (4 + 1) + 1 = 5 + 1 = 6$ by our axiom. And then again:

$4 + 3 = 4 + (2 + 1) = (4 + 2) + 1 = 6 + 1 = 7.$

Peano's axioms for multiplication are: $x \cdot 1 = x$ and $x (y + 1) = x y + x$

And for exponentiation similarly: $x^1 = x$ and $x^{y+1} = x^y x$

With these operational axioms, the previously mentioned N non standard objects are not avoided instead they become much more complicated. Peano was not aware of the non standard naturals but he was very aware of a much more important problem. Namely:
Just because we defined the operational values perfectly it doesn't mean that we can obtain from these operational axioms the properties of our operations.
The simplest example is what all children realize at learning the additions, namely that it is exchangeable in its order like $4 + 3 = 3 + 4$ or in general $x + y = y + x$.
But amazingly, though these are indeed derivable as empirical facts individually because for example both $4 + 3$ and $3 + 4$ have the 7 value, the general rule is not derivable.
Peano's heuristic discovery was a single scheme of infinite many axioms to derive rules in a similar step by step fashion as the values were. This is called induction. So if a $P(x)$ property is true for 1 and inherits from x to $x + 1$ then it is true for all x values:

$$[\ P(1) \text{ and } \forall x\,(\,P(x) \rightarrow P(\,x + 1)\,)\,] \quad \rightarrow \quad \forall x\, P(x)$$

Using this for the $P(x)$: $x + 1 = 1 + x$ property we see that :
$P(1)$ is true because $1 + 1 = 1 + 1$ and the induction step is also true because:
$x + 1 = 1 + x \rightarrow 1 + (\,x + 1\,) = (\,1 + x\,) + 1 = (\,x + 1\,) + 1.$
The first equality used the induction assumption the second the addition axiom.
We can conclude $\forall x\, P(x) = \forall x\,(\,x + 1 = 1 + x\,)$. Thus we proved exchangeability for $y = 1$.
Next we use induction to show that an incrementing rule of the first member, analogous to the addition axiom for increasing the second member is also true: $(x + 1) + y = (x + y) + 1$.
For $y =1$ the claim is trivial. Then writing $y + 1$ into y on the left, we can use the addition axiom, then use the assumption for y, use the axiom again, and get $y + 1$ in the right side too:
$(x +1) + (y + 1) = [\,(x + 1) + y\,] + 1 = [(x + y\,) + 1)\,] + 1 = [\,x + (y + 1)\,] + 1.$
Now we can regard the $R(x\,,y)$: $x + y = y + x$ relation as a rule to be proved.
We use again property induction for $P(x) = \forall y\, R(x\,,y)$. That is:

$$[\ \forall y\, R(1\,,y) \text{ and } \forall x\,(\,\forall y\, R(x\,,y) \rightarrow \forall y\, R(x + 1\,,y)\,)\,] \quad \rightarrow \quad \forall x \forall y\, R(x\,,y)$$

$R(1\,,y)$ is $1 + y = y + 1$ so this is true by what we proved above by simple letter alteration.

The induction step can be seen as follows:
$\forall y\,(\,x + y = y + x\,) \rightarrow \forall y\,(\,(x + 1) + y = (x + y) + 1\,) = (y + x) + 1 = y + (x + 1)\,)$
The first equality used the above proved first member incrementing rule, the second used the induction assumption, the third used the addition axiom with exchanged letters.
As we see, to prove even this simplest rule, we needed many tricky uses of induction.
The big question is whether induction is enough to prove all truths about the operations.
A major result of Gödel proved that if multiplication is allowed then there will be truths unprovable from any scheme of axioms, so induction can not prove every truth either.
But this was a theoretical result and didn't relate to any known classical unproved claims.
The most infamous classical open problem was Fermat's Last or rather "Lost" Theorem.
The claim is that for larger than 2 powers, $x^n + y^n \neq z^n$.
Euler the greatest mathematician before Gauss, almost proved the simplest $n = 3$ case.
But even this required very tricky induction and relied on some assumptions that he didn't quite prove. Gauss went a different way for $n = 3$ but even he couldn't solve the general claim.
Today we still don't know whether an inductive proof is possible for all n values.

Our subject is an even simpler, yet unsolved claim.
We don't need exponentiation, rather introduce a new operation.
This will depend on three variables $d\,,m\,,s$ standing for divider, multiplier and shift.
So, d and m are naturals but s can be negative too.
Our new $x\,(:d \text{ or} \cdot m + s)\,y$ operation is similar to the normal operations in the sense that y is an application number but now the x is not repeated rather it is only an initial value.

So: $x\,(:d \text{ or} \cdot m + s)\ 1 \;=\; x$. Then the bracket actually tells what to do:

We try to divide by d. If this is possible then:

$$x\,(:d \text{ or} \cdot m + s)\ (y+1) \;=\; \frac{x\,(:d \text{ or} \cdot m + s)\ y}{d}\,.$$

If $x\,(:d \text{ or } \cdot m + s)\ y$ is not dividable by d then:

$$x\,(:d \text{ or} \cdot m + s)\ (y+1) \;=\; [\, x\,(:d \text{ or} \cdot m + s)\ y \,]\ m + s.$$

If we list the possible values for a fix x and $y = 1, 2, 3, \ . \ . \ . \ .$ then this sequence is denoted as $\{:d \text{ or} \cdot m + s\}(x)$. For example:

$\{:3 \text{ or} \cdot 2+1\}(1) \;=\; 1, 3, 1, 3, 1, \ldots$

$\{:3 \text{ or} \cdot 2+1\}(2) \;=\; 2, 5, 11, 23, 47, \ldots$

Observe that here at the second case the remainder to 3 stays 2.
Also observe that if $m = d$ then such fix remainder comes about from any start.
So only the $m \neq d$ cases are really interesting but even for these the remainders are important and so we'll list their sequences too, using $< >$ brackets instead of the $\{\ \}$ brackets.
For the above two sequences:

$< :3 \text{ or} \cdot 2+1>(1) \;=\; 1,\ 0,\ 1,\ 0,\ 1,\ \ldots$
$< :3 \text{ or} \cdot 2+1>(2) \;=\; 2,\ 2,\ 2,\ 2,\ 2,\ \ldots$

This might give the impression that other initial values give the same sequences because the remainders to 3 can only be 0 , 1 , 2. But this is not quite true!
It is true for any sequence that the non 0 remainders determine the next remainder but the 0 ones imply division and the result can have any new remainder.
To see this, lets check out the 100 initial valued sequence of the above case:

$\{:3 \text{ or} \cdot 2+1\}(100) \;:\; 100,\ 201,\ 67,\ 135,\ 45,\ 15,\ 5,\ 11,\ 23,\ \ldots$
$< :3 \text{ or} \cdot 2+1>(100) :\; 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 2,\ 2,\ 2,\ \ldots$
So we entered from a 0 remainder into our 2 cycle.

The non 0 remainders that eventually step into a 0 remainder, form a single path while other non 0 remainders can form cycles. But we may not have cycles at all.
Above, the 0 path or simply the "path" is (1 , 0) and we have one cycle (2).

As next example, we should check out the other simplest interesting, that is $m \neq d$ case than the above 3 , 2 , 1 choice. This is 2 , 3 , 1. Again from the 1 , 2 , 100 initial values:

$\{:2 \text{ or} \cdot 3+1\}(1) \;=\; 1,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ \ldots$
$< :2 \text{ or} \cdot 3+1>(1) \;=\; 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ \ldots$

$\{:2 \text{ or} \cdot 3+1\}(2) \;=\; 2,\ 1,\ 4,\ 2,\ 1,\ \ldots$
$< :2 \text{ or} \cdot 3+1>(2) \;=\; 0,\ 1,\ 0,\ 0,\ 1,\ \ldots$

$\{:2 \text{ or} \cdot 3+1\}(100) = 100,\ 50,\ 25,\ 76,\ 38,\ 19,\ 88,\ 44,\ 22,\ 11,\ 34,\ 17,\ 52,\ 26,$
$13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ \ldots$
$< :2 \text{ or} \cdot 3+1>(100) = 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,$
$0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ \ldots$

Here we have no cycles only the path (1 , 0) and yet this case is much more complex than the previous. A general reason is simply that if we have cycles then the unpredictable remainders after the 0 remainders eventually enter a cycle by chance. But for our concrete first case we can be much more specific too. Indeed, observe that if a z number has 1 remainder to 3 then $2z + 1$ will be dividable by 3 but hen $\frac{2z+1}{3} < z$ except if $z = 1$.

So every x initial value creates a sequence that either has decreasing 1 remainder values that reach the bottom 1 and so we enter the 1 , 3 alteration sequence, or we get a 2 remainder and then it stays, that is we enter the (2) cycle.

The 2 , 3 , 1 case has only two possible remainders 0 or 1. Even or odd members.
After an odd we always have an even and this must be halved as many times as possible.
But now $\frac{3z+1}{2} > z$ and so the odd values are not necessarily decreasing.
Indeed, above in the 100 sequence the odds are 25 , 19 , 11 , 17 , 13 , 5 , 1.
So we had an increase after 11. Thus, there is no guarantee that a sequence will get into 1.
And yet experimentally we always see this. This claim is the Collatz Conjecture.
Formally: $\forall x \exists y \;\; x\,(:2 \text{ or} \cdot 3{+}1)\,y = 1$.
Or with our sequence notation: $\forall x \; [\, 1 \in \{:2 \text{ or} \cdot 3{+}1\}(x) \,]$.

Now I show an example with two cycles:

$\{:5 \text{ or} \cdot 4+1\}(1) = 5 \,,\, 1 \,,\, 5 \,,\, \dots$
$< :5 \text{ or} \cdot 4+1 >(1) = 0 \,,\, 1 \,,\, 0 \,,\, \dots$
$\{:5 \text{ or} \cdot 4+1\}(1) = 9 \,,\, 37 \,,\, 149 \,,\, \dots$
$< :5 \text{ or} \cdot 4+1 >(2) = 4 \,,\, 2 \,,\, 4 \,,\, \dots$

We only have the 0 , 1 , 2 , 4 remainders so far, so 3 is still missing and indeed:

$\{:5 \text{ or} \cdot 4+1\}(3) = 13 \,,\, 53 \,,\, \dots$
$< :5 \text{ or} \cdot 4+1 >(3) = 3 \,,\, 3 \,,\, \dots$

So we have the path (1 , 0) and two cycles: (2 , 4) and (3).
Here is a list of the basic $:d \text{ or} \cdot m+1$ paths and cycles:

$:2 \text{ or} \cdot 3+1$: (1 , 0)

$:2 \text{ or} \cdot 5+1$: (1 , 0)

$:3 \text{ or} \cdot 2+1$: (1 , 0) , (2)

$:3 \text{ or} \cdot 4+1$: (1 , 2 , 0)

$:3 \text{ or} \cdot 5+1$: (1 , 0) , (2)

$:4 \text{ or} \cdot 3+1$: (1 , 0) , (2 , 3)

$:5 \text{ or} \cdot 2+1$: (1 , 3 , 2 , 0) , (4)

$:5 \text{ or} \cdot 3+1$: (1 , 4 , 3 , 0) , (2)

$:5 \text{ or} \cdot 4+1$: (1 , 0) , (2 , 4) , (3)

$:7 \text{ or} \cdot 5+1$: (1 , 6 , 3 , 2 , 4 , 0) , (5)

As I said, the strange thing is that those that have cycles are the simpler because their sequences always eventually end in that. The three that have no cycles only the path, are uncertain for ever.

But some chance arguments can be applied as follows:

We only should regard the non dividable cases, namely how much decrease or increase can happen till the next non dividable is reached.

The $d-1$ many possible non 0 remainders give all different $m^{d-1}, m^{d-2}, \ldots, m$ multiplications depending on how many steps we will do in our path. Then they all go into dividability which means $1, 2, 3, \ldots$ many possible divisions by d.
Exactly one dividability is in the $\frac{d-1}{d}$ portion of dividables.
Then the exactly twice dividables are again $\frac{d-1}{d}$ portion of the leftovers $1-\frac{d-1}{d} = \frac{1}{d}$, that is : $\frac{d-1}{d^2}$. Similarly, the exactly three times dividables are $\frac{d-1}{d}$ portion of

$$1-\frac{d-1}{d}-\frac{d-1}{d^2} = \frac{d^2-d^2+d-d+1}{d^2} = \frac{1}{d^2} \text{ so they are } \frac{d-1}{d^3} \text{ in portion.}$$

These $\frac{d-1}{d} + \frac{d-1}{d^2} + \frac{d-1}{d^3} + \ldots$ portions together should be 1 and indeed they are:

$$\frac{d-1}{d}\left(1+\frac{1}{d}+\frac{1}{d^2}+\ldots\right) = \frac{d-1}{d}\frac{1}{1-\frac{1}{d}} = 1.$$

But now we use these as weighs to apply the corresponding d divisions.
The possible m multiplications can be applied prior:

$$(m^{d-1}+m^{d-2}+\ldots+m)\left(\frac{d-1}{d}\frac{1}{d}+\frac{d-1}{d^2}\frac{1}{d^2}+\frac{d-1}{d^3}\frac{1}{d^3}+\ldots\right) =$$

$$(m^{d-1}+m^{d-2}+\ldots+m)\frac{d-1}{d^2}\left(1+\frac{1}{d^2}+\frac{1}{d^4}+\frac{1}{d^6}+\ldots\right) =$$

$$(m^{d-1}+m^{d-2}+\ldots+m)\frac{d-1}{d^2}\frac{1}{1-\frac{1}{d^2}} = (m^{d-1}+m^{d-2}+\ldots+m)\frac{1}{d+1}.$$

For $:2$ or $\cdot 3+1$ we get $3 \cdot \frac{1}{2+1} = 1$ so we are at limbo.
But we don't have infinite many dividabilities so this brings back the sequence from the brink.

For $:2$ or $\cdot 5+1$ we get $5 \cdot \frac{1}{2+1} = \frac{5}{3}$ so the sequence should grow in average.

For $:3$ or $\cdot 4+1$ we get $(16+4) \cdot \frac{1}{3+1} = 5$ so the sequence usually grows very fast.

$:2$ or $\cdot 5+1$ will return to earlier values from the $x = 1, \ldots, 8$ starting values.
But from $x = 9$ it runs off to infinity:

$\{:2 \text{ or} \cdot 5+1\}(9) =$

46 ,	23 ,	116 ,	58 ,	29 ,	146 ,	73 ,	366 ,	183 ,	916 ,	458 ,	229 ,	1146 ,	573 ,	2866 ,	1433 ,	7166 ,	. .
0	1	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	

The second already runs off from $x = 1$:

$\{:3 \text{ or} \cdot 4+1\}(1) =$

1 ,	5 ,	21 ,	7 ,	29 ,	117 ,	39 ,	13 ,	53 ,	213 ,	71 ,	285 ,	95 ,	381 ,
1	2	0	1	2	0	0	1	2	0	2	0	2	0

127 ,	509 ,	2037 ,	679 ,	2717 ,	10869 ,	3623 ,	14493 ,	4831 ,	19325 ,	77301 ,
1	2	0	1	2	0	2	0	1	2	0

25767 ,	8589 ,	2863 ,	11453 ,	45813 ,	15271 ,	61085 ,	. . .
0	0	1	2	0	1	2	

As I said, the $:2 \text{ or} \cdot 3+1$ Collatz sequences are at limbo and yet they always return to 1.
The probabilistic limbo is the same for $:2 \text{ or} \cdot 3-1$ but here we have more possible cycles beside the 1 minimal valued. But again, all sequences enter these cycles.
The exact situation is that every sequence encounters a value lower or equal than the start.
Thus, all sequences return to earlier and earlier ones till finally a returning one repeats.
We don’t know exactly these returning start values, neither can we prove this claim.
Even initial values trivially become smaller at once. Here are the odd starts with the sequence beginnings up to a lower odd, or a return to the start, that is forming a cycle first time:

1 , 2 , 1

3 , 8 , 4 , 2 , 1

5 , 14 , 7 , 20 , 10 , 5

7 , 20 , 10 , 5

9 , 26 , 13 , 38 , 19 , 56 , 28 , 14 , 7

11 , 32 , 16 , 8 , 4 , 2 , 1

13 , 38 , 19 , 56 , 28 , 14 , 7

15 , 44 , 22 , 11

17 , 50 , 25 , 74 , 37 , 110 , 55 , 164 , 82 , 41 , 122 , 61 , 182 , 91 , 272 , 136 , 68 , 34 , 17

As we see, we found three cycles with minimal values 1 , 5 , 17.