Farey Sequences

History

In a women's magazine in the middle of the 19-th century, the question was raised to find the number of different fraction values among the fractions with denominators under 100.
The solution came very slowly but more generally by finding the number of elements in the S_d set of all totally simplified fractions under 1 with denominators up to d.
By the way we'll call the members of S_d "simply" as simple fractions.
Anyway, the answer to the original question is then the number of elements of S_{99}.
Farey ignored the original problem and instead regarded the strictly increasing orders of S_d as F_d sequences :

$$F_1 = \frac{0}{1} < \frac{1}{1}$$

$$F_2 = \frac{0}{1} < \frac{1}{2} < \frac{1}{1}$$

$$F_3 = \frac{0}{1} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{1}{1}$$

$$F_4 = \frac{0}{1} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{1}{1}$$

$$F_5 = \frac{0}{1} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{1}{1}$$

$$F_6 = \frac{0}{1} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \frac{1}{1}$$

And he observed that in every such sequence for any three consecutive members $\frac{a}{b} < \frac{n}{d} < \frac{A}{B}$ we have that $\frac{n}{d} = \frac{a+A}{b+B}$. So the middle member is a "dumb-sum" of the two neighboring ones.
He couldn't prove this and indeed it would be almost impossible to do this for an amateur.
But he actually missed a simpler law that is visible in the sequences too and which would have directed him towards a proof of his observation.
Namely, that for the two $\frac{a}{b} < \frac{n}{d}$ and $\frac{n}{d} < \frac{A}{B}$ neighboring members:

$$\frac{n}{d} - \frac{a}{b} = \frac{nb - da}{db} = \frac{1}{db} \quad \text{and} \quad \frac{A}{B} - \frac{n}{d} = \frac{Ad - Bn}{Bd} = \frac{1}{Bd}.$$

From $nb - da = 1$ and $Ad - Bn = 1$ by subtraction we get that $nb + Bn = da + Ad$ and so $n (b + B) = d (a + A)$ which is exactly what he discovered.

So if we could show that for any neighboring $\frac{n}{d} < \frac{n'}{d'}$ members $n'd - nd' = 1$ which we'll call as being a Bezout pair of fractions, then we are finished.

The Plan

We can narrow our claim by calling in F_d the $\frac{a}{b} < \frac{n}{d}$ or $\frac{n}{d} < \frac{A}{B}$ neighbors principal.
Then it would be enough to prove our claim for principal neighbors because any neighbors are principals in the first Farey sequence that contains both of them.
But strangely, we'll go back to the idea of triplets too and so we'll call in F_d the $\frac{a}{b} < \frac{n}{d} < \frac{A}{B}$ consecutive triplets as principal too.
Observe that for such $\frac{a}{b} < \frac{n}{d} < \frac{A}{B}$ principal triplets, obviously $\frac{a}{b} < \frac{A}{B}$ are neighbors in all earlier Farey sequences where they both appear. And this rings a bell for using induction.
If assuming that $\frac{a}{b} < \frac{A}{B}$ is a Bezout pair implies that $\frac{a}{b} < \frac{n}{d}$ and $\frac{n}{d} < \frac{A}{B}$ are Bezout pairs in all sequences where $\frac{a}{b} < \frac{n}{d} < \frac{A}{B}$ are principal then we are finished.

This plan started from a specialization of the neighbor relation but we can generalize it too:
Two members can be called consecutive if they are neighbors or all the members between them have bigger denominators than theirs.
This second means that again in the first Farey sequence that contains both they were principal neighbors and so we at once realize that:
For consecutive $\frac{a}{b} < \frac{A}{B}$ members we again have that $Ab - Ba = 1$.
But we'll make an even more important leap and regard all fractions up to 1.

So any two $\frac{a}{b}, \frac{A}{B} < 1$ fractions are:
Consecutive pair if : $\frac{a}{b} < \frac{A}{B}$ and any fraction between them has denominator above b and B.

Bezout pair if : $\frac{A}{B} - \frac{a}{b} = \frac{Ab - Ba}{Bb} = \frac{1}{Bb}$.

And our claim is that: Every consecutive pair of two simple fractions is a Bezout pair.

Now I'll explain the Bezout name for our second relation. It starts with an admission.
There was a sloppiness in how the question raised in the women's magazine was replaced as the number of elements in S_{99}. Indeed, the magazine specifically asked for the number of different values and so it is only the same as the number of simple fractions if there are no non identical but equal simple fractions. This is true but not trivially.
Even more explicit was this mistake in how Farey regarded S_d as F_d trivially.
This is not mentioned in any articles treating the Farey sequences and yet as I will show, it is actually the essence of everything. Formally, we corrected the mistake by regarding all fractions up to 1. But this is not enough and the further details will explain the Bezout name.

That two totally simplified fractions can not be equal is a crucial fact of number theory that immediately gives Euclid's Lemmas and through this the Fundamental Theorem of Arithmetic.
You can find all this in my article with this title. In fact, I used a fractional approach as start.
But there is a more accepted abstract approach using Bezout's identity which claims that a greatest common divider is also a linear combination: $< a , b > = \pm \alpha a \pm \beta b$.
In particular if a, b are relative primes then $1 = \alpha a - \beta b = \beta' b - \alpha' a$.
To be variants $\frac{a}{b} = \frac{A}{B}$, means $aB - bA = 0$ so 0 linear combination.

This suggests that Bezout's 1 linear combinations should also be regarded as second fractions. And indeed, $\alpha a - \beta b = 1$ and $\beta' b - \alpha' a = 1$ means that $\frac{a}{b}, \frac{\beta}{\alpha}$ and $\frac{\beta'}{\alpha'}, \frac{a}{b}$ are Bezout pairs as we defined it above. So we should also expect this to relate to the mentioned fractional variant result that the totally simplified fractions are unique among the variants.
The proof of this went by characterizing the variants as expansions of the minimal variant that therefore is also the minimal simplification and thus the single total simplification too.
So now we can realize what is never mentioned when the amazing subject of Farey sequences is examined, namely that it is actually a characterization of the simple fractions showing from the outside why totally simplified fractions can not be equal.

But now some surprising facts about the Bezout pairs:
For a chosen $\frac{n}{d}$ simple fraction there are infinite many smaller $\frac{a}{b}$ and bigger $\frac{A}{B}$ fractions that form Bezout pair with $\frac{n}{d}$, that is $nb - da = 1$ and $dA - nB = 1$. It's easy to show that unique "smallest" pairs exist "under" $\frac{n}{d}$, that is with $a, A < n$ and $b, B < d$.
For $\frac{2}{3}$ for example these are $\frac{1}{2}$ and $\frac{1}{1}$ so the minimal Bezout triplet of $\frac{2}{3}$ is $\frac{1}{2} < \frac{2}{3} < \frac{1}{1}$.
The other triplets of $\frac{2}{3}$ can be obtained by simply "adding" $\frac{2}{3}$ repeatedly to these minimals.
But not literally, rather using the earlier mentioned "dumb-sum" observed by Farey:
So we have that: $\frac{1}{2}, \frac{3}{5}, \frac{5}{8}, \ldots \quad \rightarrow \frac{2}{3} \leftarrow \quad \ldots, \frac{5}{7}, \frac{3}{4}, \frac{1}{1}$.
Amazingly, not only the approaching members are all Bezout pairs with $\frac{2}{3}$ but the consecutive members in them are too.

These are all very exciting and easily provable but still don't prove our main claim.
A proof of that requires first to prove facts about Bezout pairs, including that they are all consecutive pairs of two simple fractions.
Using these can we only show the reversal that was our main claim.

<u>Bezout Pair Consequences:</u>

For any $\frac{a}{b} < \frac{A}{B}$ Bezout pair:

a. Both members are simple fractions.

b. The $\frac{a+A}{b+B}$ dumb-sum of them is between them, in fact:
Forms again Bezout pairs with both members.

c. $\frac{a+A}{b+B}$ is again a simple fraction.

d. An $\frac{n}{d}$ fraction between them can only be with $d \geq b + B$.
Thus a Bezout pair is a consecutive pair of two simple fractions.

e. $\frac{a+A}{b+B}$ is the single fraction between them with denominator $\leq b + B$.

Proofs:

a. $Ab - Ba = 1$ implies that there can be no $c > 1$ common divider of A , B or a , b.
Indeed, c would have to divide $Ab - Ba$. The ≤ 1 is trivial by assumption as pair.

b. $$\frac{a+A}{b+B} - \frac{a}{b} = \frac{(a+A)b - (b+B)a}{(b+B)b} = \frac{Ab - Ba}{(b+B)b} = \frac{1}{(b+B)b}$$

$$\frac{A}{B} - \frac{a+A}{b+B} = \frac{A(b+B) - B(a+A)}{B(b+B)} = \frac{Ab - Ba}{B(b+B)} = \frac{1}{B(b+B)}$$

c. By b. it is Bezout pair with $\frac{a}{b}$ and then by a. it is a simple fraction.

d. $$\frac{1}{Bb} = \frac{Ab - Ba}{Bb} = \frac{A}{B} - \frac{a}{b} = (\frac{A}{B} - \frac{n}{d}) + (\frac{n}{d} - \frac{a}{b}) = \frac{Ad - Bn}{Bd} + \frac{nb - da}{db} \geq \frac{1}{Bd} + \frac{1}{db} =$$

$$= \frac{b+B}{Bdb}$$ And so $d \geq b + B$.

e. $\frac{a+A}{b+B}$ is between them by b. Any other $\frac{n}{d}$ fraction between them is between either $\frac{a}{b}$ and $\frac{a+A}{b+B}$ or $\frac{a+A}{b+B}$ and $\frac{A}{B}$. By b. these are simple pairs and so by d.

d can't be smaller than $b + b + B$ or $b + B + B$.

The Reversal

Our main claim was the reversal of the consequence in d.
We'll prove that but first observe a few things:
The negative version of a $P \rightarrow Q$ implication is not it's reversal $Q \rightarrow P$ rather $\neg P \rightarrow \neg Q$.
The negative version of our consequence in d. for example is that if two fractions are not consecutive then they are not Bezout pair, that is $Ab - Ba \neq 1$.

The negative version of the reversal however is more interestingly different.

First of all, it says that if $\frac{a}{b} < \frac{A}{B}$ are not Bezout pair, that is $Ab - Ba \neq 1$ but they are both simple then they are not consecutive, so there is $\frac{n}{d}$ fraction between them with $d < b$ or B.
The strangeness lies in that now we don't deny an existence rather claim one.
And so, we can search the $d < b$ or B denominator fractions and experimentally verify our claim by finding a concrete fraction. For example $\frac{2}{3} < \frac{4}{5}$ are not Bezout pair and indeed in few seconds we'll find $\frac{3}{4}$ between them. Thus we feel that these concrete examples must have a rule and so we should search for it. And many times this is a good strategy for a proof.
But not here! The search for such rule is now a wild goose chase! **There is no rule!**
Of course, we will prove that such small fraction exists but not by finding one concretely.
As a practical side of all this, you have a perfect "smart ass challenge".
Indeed, if a "professional" doesn't know about the Farey fractions he'll fall for the trap and tries to find a rule for the in-between fractions.
Interestingly, a similar trap exists for the earlier mentioned Euclid's Lemma.

The usual form says that:
If a p prime divides bc then p must divide at least one of the members b or c.
But this is equivalent to that if p doesn't divide either of b , c then it can not divide bc either.
As a consequence, that actually implies the full problem:
If b , c < p then p can not divide bc. Now a further twist: Not dividing something means having non zero remainder when doing the division and so denoting this remainder with $|bc|_p$, the claim can be put as: $b, c < p \rightarrow |bc|_p \neq 0$.
Let's see the actual $|bc|_p$ values for the first few primes:

$p = 2 \rightarrow (b,c) = (1,1) \rightarrow |bc|_p = 1 \neq 0.$

$p = 3 \rightarrow (b,c) = (1,1), (1,2), (2,1), (2,2) \rightarrow |bc|_p = 1, 2, 2, 1 \neq 0.$

$p = 5 \rightarrow (b,c) = (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3), (4,4) \rightarrow |bc|_p = 1, 2, 3, 4, 2, 4, 1, 3, 3, 1, 2, 4, 3, 2, 1.$

As we see, the remainders are totally irregular but "magically" never become 0.
But Euclid's Lemma has a further twist. Namely, we can now apply the negated form and say:
If there are b , c < a numbers that $|bc|_a = 0$ then a can not be a prime. More amazingly:
If for a b there is such c and c_1 is the first, then this is actually a non trivial divider of a.
So the effective counterexample has a role in the proof. Here we don't have such continuation!
The reason could be that here the condition itself is not really that effective!
Indeed, an observed $Ab - Ba \neq 1$ requires to search all the simple fractions with lower denominators but simplicity is not easy to establish for large values.
A further twist is that $Ab - Ba \neq 1$ means $Ab - Ba > 1$ or $Ab - Ba = 0$ or $Ab - Ba < 0$.
So strangely, the minimal variant existence is hidden in the search too!!!
All these show a still not explored side of mathematics.
So we'll just prove the reversal but we need some preparation to make the proof even cleaner.

Minimal consecutives:

1. Observe that if a < A then for fractions under 1 we have that $\frac{a}{b} < \frac{a}{b-1} < \frac{A}{b}$.

Indeed, $ab - a < ab < bA - A$. The first < is trivial and for the second observe that:
$A < b \leq b(A - a) = bA - ab$ which implies the second.
This implies that same denominator fractions can not be consecutive.
To an $\frac{n}{d}$ fraction there can be no two fractions under or above both having denominators under n and both being consecutive to $\frac{n}{d}$. This is trivial by the definition of consecutives.
But if d > 1 we obviously have lower denominator fractions both under and above $\frac{n}{d}$.
By their finite number and the previous observation thus we have single lower denominator consecutives to $\frac{n}{d}$ under and above which we call minimal and denote as $\left(\frac{n}{d}\right)^-$ and $\left(\frac{n}{d}\right)^+$.

2. If in a consecutive pair a member is not simple then simplifying it the pair remains consecutive. Thus the previously defined minimal consecutives of an $\frac{n}{d}$ are always simple.

If $\frac{n}{d}$ itself is simple then its minimal consecutives are the neighbors of $\frac{n}{d}$ in F_d.
These are what we called earlier as principal triplets. Now a more appropriate name would be minimal consecutive triplet. At any rate, the two sides are obviously a consecutive pair and so:

The minimal consecutives of a simple fraction are a consecutive pair of simple fractions.

Induction Lemma:

If $\left(\frac{n}{d}\right)^{-} = \frac{a}{b}$ and $\left(\frac{n}{d}\right)^{+} = \frac{A}{B}$ are Bezout pair then $\frac{n}{d} \equiv \frac{a+A}{b+B}$, that is:
$n = a + A$ and $d = b + B$.

Proof:

$\frac{n}{d}$ is between $\frac{a}{b} < \frac{A}{B}$ and so by d. of the Bezout pair consequences $d < b + B$ is impossible.
$d > b + B$ is impossible too by b. because $\frac{a+A}{b+B}$ were already an in-between fraction under d denominator so it would be between either $\frac{a}{b}$ and $\frac{n}{d}$ or $\frac{n}{d}$ and $\frac{A}{B}$ contradicting how we defined these. So $d = b + B$ and then by e. $n = a + A$ too.

Reversal Theorem:

Every consecutive pair of two simple fractions is a Bezout pair.

Proof:

We use induction on the denominators.
The start is $d = 1$ where the claim is trivial because $\frac{0}{1}$, $\frac{1}{1}$ are the only fractions.
They are a consecutive pair of simple fractions and Bezout pair too.

Now suppose our claim is true for all b , $B < d$ denominator $\frac{a}{b}$ and $\frac{A}{B}$ fractions, that is if these are simple and are a consecutive pair then they are Bezout pair too.

We'll show for all consecutive fractions with up to d denominators that they are Bezout pairs.
If both fractions have denominators under d then of course the induction assumption implies it.
If one of them is $\frac{n}{d}$ then the other must have smaller denominator and be a minimal consecutive pair of $\frac{n}{d}$.
Regarding the other minimal consecutive pair of $\frac{n}{d}$, that is the $\frac{a}{b} < \frac{n}{d} < \frac{A}{B}$ minimal consecutive triplet, then enough to prove that $\frac{a}{b} < \frac{n}{d}$ and $\frac{n}{d} < \frac{A}{B}$ are Bezout pairs.
We'll use our previous lemma. There we needed as assumption that $\frac{a}{b} < \frac{A}{B}$ are Bezout pair.
But now this follows from our induction assumption because these are a consecutive pair of simple fractions with denominators under n. So using the lemma we get that $\frac{n}{d} = \frac{a+A}{b+B}$.

And then by b. $\frac{a}{b} < \frac{n}{d}$ and $\frac{n}{d} < \frac{A}{B}$ are both Bezout pairs.