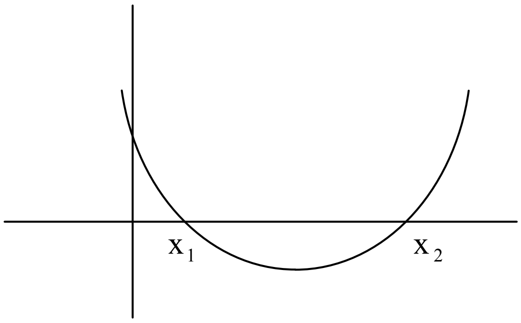
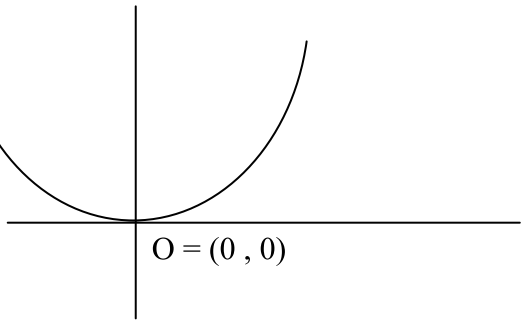
1. Second Order Equation

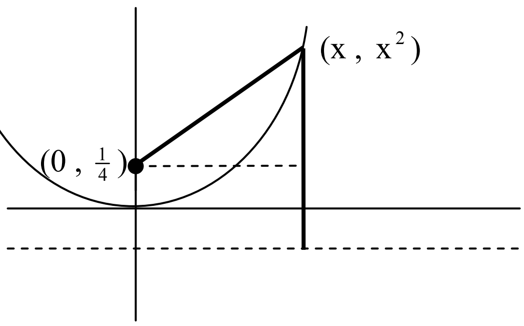
The second order $a\,x^2 + b\,x + c = 0$ equations are taught in every high school.
The real importance is that many word problems lead to such.
Unfortunately, instead it became a useless monstrosity on its own.
Since the coordinate system is also part of every high school math curriculum, thus the $y = a\,x^2 + b\,x + c$ function or so called parabola is also mentioned.
These two relate by the fact that the x_1, x_2 roots of the second order equation are the crossings of the parabola with the x axis:



The position and fatness or narrowness of the parabola depend on a , b , c and it's easy to see that if a is negative, then the parabola is upside down. It's harder to tell from a , b , c whether a crossing with the x axis is happening or not.
The simplest parabola is $y = x^2$ that has the single root at the origin:



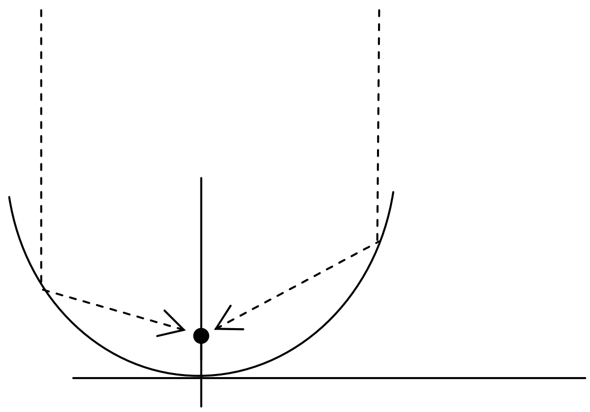
This parabola is interesting on its own geometrically too because it is the collection of those points that are equal distanced from the $(0 , \frac{1}{4})$ point and the $y = -\frac{1}{4}$ line:



Indeed, the distance of (x , x^2) from the line is $x^2 + \frac{1}{4}$ and from the $(0 , \frac{1}{4})$ point by the Pythagoras Theorem is:

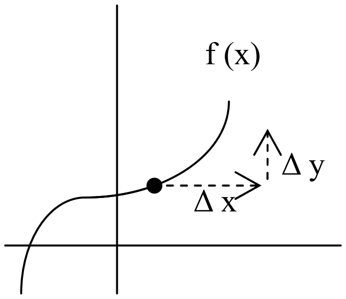
$$\sqrt{x^2 + (x^2 - \frac{1}{4})^2} = \sqrt{x^2 + x^4 - \frac{x^2}{2} + \frac{1}{16}} = \sqrt{x^4 + \frac{x^2}{2} + \frac{1}{16}} = \sqrt{(x^2 + \frac{1}{4})^2} = x^2 + \frac{1}{4}$$

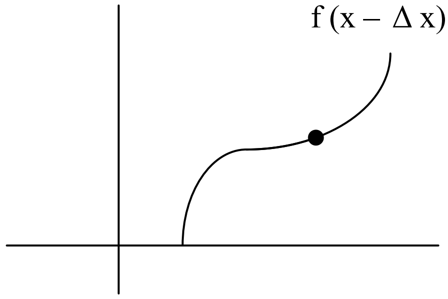
A second more important feature is that this same $(0 , \frac{1}{4})$ point is a focal point, meaning that all vertical light rays coming into the parabola are reflected into here:



This is more than geometrically important because it means that for astronomical objects that are practically infinite distanced, a parabolic mirror is the perfect "reflector".
This is the principle of the Newtonian reflecting telescope.
The mirror is a turned around surface from the bottom of a parabola.
The chosen units determine how curved the mirror will be that is how far the $\frac{1}{4}$ focal point is.
Amazingly, a pot of mercury if spun around will also form such parabolic surface and then the focus that is the curvature is depending on how fast we spin the pot.
Finally, the most important use of parabola is in ballistics.
The reason is that if we regard a local flat surface of the earth with fix gravity above, then objects accelerate with fix vertical value and this gives a distance increasing with squared time.
Amazingly, the horizontal speeds are independent of gravity and so neglecting the air resistance thus we get fix speed motions horizontally. The two directions combined then give parabolic motions. This is how canon bullets would travel without the air corrections.

The examination of the general $y = a x^2 + b x + c$ parabola should also start from the basic.
The general fundamental idea is how to shift an arbitrary $y = f(x)$ function Δx horizontally and Δy vertically:

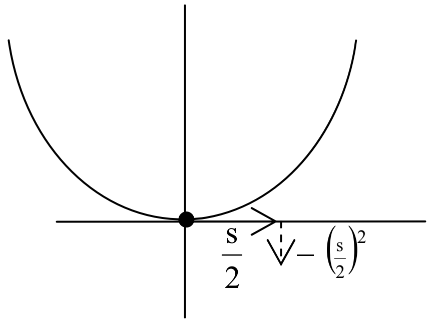


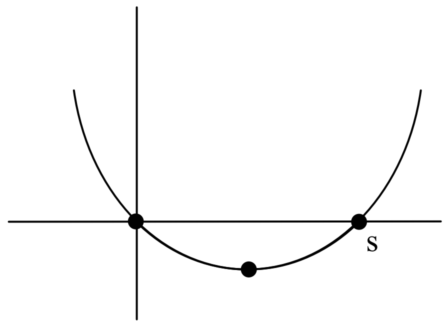


Then shifting the normal $y = x^2$ parabola with $\Delta x = \frac{s}{2}$ and $\Delta y = -\left(\frac{s}{2}\right)^2$ becomes

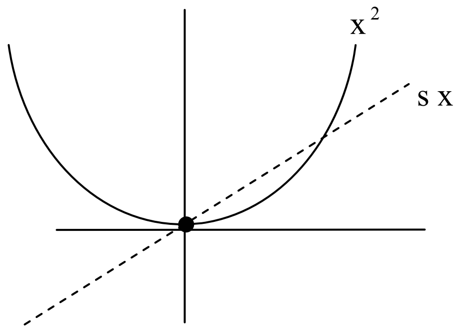
$$(x - \frac{s}{2})^2 - \left(\frac{s}{2}\right)^2 = x^2 - s x + \left(\frac{s}{2}\right)^2 - \left(\frac{s}{2}\right)^2 = x^2 - s x .$$

So this has $x_1 = 0$ and $x_2 = s$ as "second" root:





The fact that it is a mere shifted version of the normal parabola is surprising because the $y = x^2 - s\,x$ equation would suggest that the symmetrical normal parabola is distorted by subtracting the $y = s\,x$ line which by the way has s as "slope":

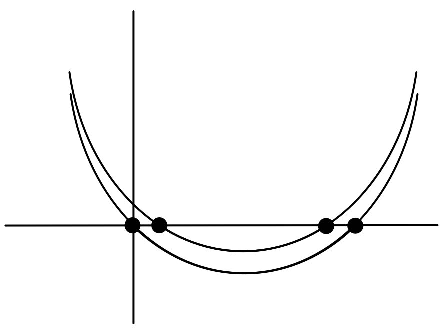


The previous two meanings of s as "second" root or "slope" are merely coincidental and now we show the real third meaning:

For this we start with the general $a\,x^2 + b\,x + c$ equation and try to force it into our previous $(x - \frac{s}{2})^2 - \left(\frac{s}{2}\right)^2 = x^2 - s\,x$ form:

$$a\,(x^2 + \frac{b}{a}\,x + \frac{c}{a}\,) = a\,(x^2 - s\,x + p\,) = a\,\left[\,(x - \frac{s}{2})^2 - \left(\frac{s}{2}\right)^2 + p\,\right].$$

So we indeed succeeded with the horizontal and vertical shifts plus a p extra vertical shift inside the square bracket. Then the single a multiplier as a vertical stretcher is used.
This final stretch keeps the roots and so our only problem is that the inside extra p vertical shift ruins the beautiful $x_1 = 0$ and $x_2 = s$ roots:



To find the altered roots is easy though algebraically:

$$(x - \frac{s}{2})^2 - \left(\frac{s}{2}\right)^2 + p = 0$$

$$(x - \frac{s}{2})^2 = \left(\frac{s}{2}\right)^2 - p$$

We see at once that if $\left(\frac{s}{2}\right)^2 - p$ is negative then there can be no x solution because the left can not be negative only 0 or positive.

If $\left(\frac{s}{2}\right)^2 - p = 0$ then $x = \frac{s}{2}$ is single root.

If $\left(\frac{s}{2}\right)^2 - p > 0$ then the square root of this that is $\sqrt{\left(\frac{s}{2}\right)^2 - p} = \sqrt{\frac{s^2 - 4p}{4}} = \frac{\sqrt{s^2 - 4p}}{2}$ is a positive number and so $x - \frac{s}{2}$ must be this or its negative.

Thus $x = \frac{s}{2} \pm \frac{\sqrt{s^2 - 4p}}{2}$

Observe that the sum of the two roots is :

$$\frac{s}{2} + \frac{\sqrt{s^2 - 4p}}{2} + \frac{s}{2} - \frac{\sqrt{s^2 - 4p}}{2} = s$$

So the important third meaning of s is that it's the “sum” of the roots.

From this we might guess that p stood not for the plus shift, rather for the “product” of the roots and indeed:

$$\left(\frac{s}{2} + \frac{\sqrt{s^2 - 4p}}{2}\right)\left(\frac{s}{2} - \frac{\sqrt{s^2 - 4p}}{2}\right) = \left(\frac{s}{2}\right)^2 - \left(\frac{\sqrt{s^2 - 4p}}{2}\right)^2 = \frac{s^2}{4} - \frac{s^2 - 4p}{4} = p$$

The $x_1 + x_2 = s$ and $x_1 x_2 = p$ root equations might suggest that they could give the roots somehow easier. But expressing x_2 we get $x_2 = s - x_1$ and writing this into the other equation we get $x_1 (s - x_1) = p$ and so $s x_1 - x_1{}^2 = p$ from which we get $x_1{}^2 - s\, x_1 + p = 0$. So we are back to “square one”.

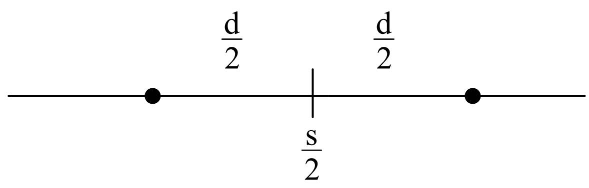
In spite of this failure, the root equations are useful.
A first use is to guess the roots.

For example in $x^2 - 7x + 12 = 0$ $x_1 + x_2 = 7$, $x_1 x_2 = 12$ so $x_{12} = 3, 4$.

Of course, for any reasonably hard example, such guessing is hopeless.

A second didactical use can be obtained by contemplating that given the sum and product of two numbers, the third logical missing data should be their difference $d = |x_1 - x_2|$.

First of all, if we knew this then we at once knew the x_1 and x_2 roots too because $\frac{s}{2}$ is the middle point of x_1 and x_2 and $\frac{d}{2}$ is their distance from this middle:



Thus: $\frac{s}{2} + \frac{d}{2}$ = the bigger of x_1 and x_2 and $\frac{s}{2} - \frac{d}{2}$ = the smaller of x_1 and x_2.

Or in short, the roots are : $\frac{s}{2} \pm \frac{d}{2}$.

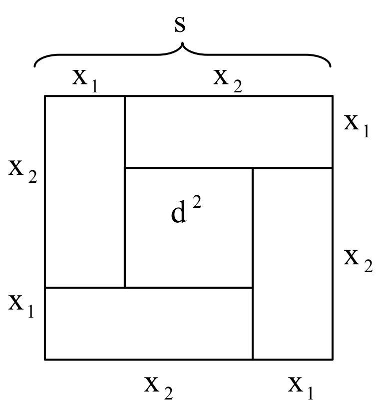
Most amazingly, this d can be obtained from s and p as $d = \sqrt{s^2 - 4p}$.
Indeed, algebraically:

$$(x_1 + x_2)^2 = s^2 = x_1^2 + x_2^2 + 2x_1x_2$$

$$(x_1 - x_2)^2 = d^2 = x_1^2 + x_2^2 - 2x_1x_2$$

So $$s^2 - d^2 = 4x_1x_2 = 4p$$

But also geometrically, the Babylonians came up with a Hindu proof very closely resembling the Pythagoras one:



$$d^2 = s^2 - 4x_1x_2 = s^2 - 4p$$

So the roots are $x = \frac{s}{2} \pm \frac{d}{2} = \frac{s}{2} \pm \frac{\sqrt{s^2 - 4p}}{2}$ again, but now with visual meanings behind all parts of this formula.

Finally, there is a third use of the root equations, namely to see that:

$$x^2 - sx + p = (x - x_1)(x - x_2) \quad \text{for all } x \text{ values.}$$

That is, a second order function always can be written as a product if there are roots.
This includes $x_1 = x_2$.

To see this, we simply multiply the right side and it becomes $x^2 - (x_1 + x_2)\,x + x_1 x_2$ which is exactly the left.

Back in the sixties when the second order formula was really used for word problems, a frequent dilemma was whether the two solutions are both real solutions of the problem or not. Sometimes for example, one became positive, the other negative for a distance problem that had to be positive. So it was just accepted that a meaningless extra solution can come about too.
A deeper dilemma was why $\sqrt{s^2 - 4p}$ can become meaningless already if we get negative under the square root.
But even more puzzling situations can appear for the solution formula of the third order equation with more negative parts under different square roots. They can cancel each other and so meaningful results could come out, even with such impossible parts.
This cluelessness, went on for hundreds of years.

It was assumed, that the "root" of this square root problem is merely $\sqrt{-1}$.

Indeed, by the rule $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ we have $\sqrt{-b} = \sqrt{(-1)\,b} = \sqrt{-1}\,\sqrt{b}$.

The only exact fact we know about this "imaginary" number $\sqrt{-1}$, is that its square is -1.
Or in other words: $\sqrt{-1}\,\sqrt{-1} = -1$. But already this can be contradictory!
Indeed, if we try to use exactly the above rule of $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$, then:

$$\sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

So, the i abbreviation for $\sqrt{-1}$ was introduced to avoid this and it also stood for imaginary.
Its rule is the same : $i\,i = i^2 = -1$ and yet with this i , the contradiction can be resolved:
The root of why the old $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ rule is false is that a square root is not unique.
In particular, $\sqrt{-1}$ is not only i but $-i$ too. Indeed $(-i)^2 = i^2 = -1$.

$\sqrt{-1}$ or i can be involved in all kinds of complicated expressions and it wasn't at once narrowed down to the final $u + v\,i$ dual combinations as fundamental.

In fact, the second order formula we used above is also very well hiding this form when

$b^2 - 4\,a\,c < 0$. We should use $-(4ac - b^2) = -1\ (4ac - b^2)$ to take the i out:

$$x_{12} = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a} = \frac{-b}{2\,a} \pm \frac{\sqrt{b^2 - 4\,a\,c}}{2\,a} = \frac{-b}{2\,a} \pm \frac{\sqrt{4\,a\,c - b^2}}{2\,a}\,i$$

As we see, the roots became $u \pm v\,i$ pairs of the dual combinations.

2. Algebraic Roots

The real hint towards these $u + v\,i$ dual forms as basic units, and their crucial appearance in these $u \pm v\,i$ so called "conjugate" pairs came with the recognition that they appear as roots of any algebraic equation :

$A(x) = c_n\ x^n + c_{n-1}\ x^{n-1} + \ldots + c_1\,x + c_0 = 0$. The c-s are called the coefficients.

The most educational is first to calculate just the third power of $u + v\,i$ and $u - v\,i$.
$(u + v\,i)^3 = (u + v\,i)(u + v\,i)(u + v\,i) = u^3 + 3\,u^2 v\,i + 3\,u\,(v\,i)^2 + (v\,i)^3$
Indeed, there are three ways $u, u, v\,i$ can be picked, and also three ways $u, v\,i, v\,i$.
At $(u - v\,i)^3$ only the single $(-v\,i)$ and the $u, u, -v\,i$ give negative products, so
$(u - v\,i)^3 = (u - v\,i)(u - v\,i)(u - v\,i) = u^3 - 3\,u^2 v\,i + 3\,u\,(v\,i)^2 - (v\,i)^3$.
Using $i^2 = -1$ and $i^3 = i^2\,i = -1\,i = -i$.
$(u + v\,i)^3 = (u^3 - 3\,u\,v^2) + (3\,u^2 v - v^3)\,i$ and $(u - v\,i)^3 = (u^3 - 3\,u\,v^2) - (3\,u^2 v - v^3)\,i$
So the conjugate $u + v\,i, u - v\,i$ pairs gave conjugate values.
This goes similarly for any powers, their multiples and sums too. So the end result is that:

Algebraic $A(x) = c_n\ x^n + \ldots + c_0$ gives conjugate values for conjugate $u + v\,i, u - v\,i$.

Now if $u + v\,i$ is a formal root of $A(x)$, that is:
$A(u + v\,i) = (\qquad) + (\qquad)\,i = 0$ then both $(\qquad)$ have to be 0. Thus:
$A(u - v\,i) = (\qquad) - (\qquad)\,i = 0$ too.

Beside this grand rule of symmetry, there was an equally grand formal process known for algebraic expressions, namely the exact copy of division among numbers.
The trick is just to regard the first coefficients as we proceed, member by member:

$$(3\,x^5 + 2\,x^4 - x^3 + 3\,x^2 + 5) : (2\,x^2 + x - 1) = \frac{3}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{25}{16}$$

$$\begin{array}{l}
3\,x^5 + \frac{3}{2}x^4 - \frac{3}{2}x^3 \\
\hline
0 \quad \frac{1}{2}x^4 + \frac{1}{2}x^3 + 3\,x^2 \\
\quad\ \ \frac{1}{2}x^4 + \frac{1}{4}x^3 - \frac{1}{4}x^2 \\
\hline
\qquad 0 \quad \frac{1}{4}x^3 + \frac{13}{4}x^2 + 0\,x \\
\qquad\quad\ \ \frac{1}{4}x^3 + \frac{1}{8}x^2 - \frac{1}{8}x \\
\hline
\qquad\qquad 0 \quad \frac{25}{8}x^2 + \frac{1}{8}x + 5 \\
\qquad\qquad\quad\ \ \frac{25}{8}x^2 + \frac{25}{16}x - \frac{25}{16} \\
\hline
\qquad\qquad\quad 0 \quad -\frac{23}{16}x + \frac{105}{16}
\end{array}$$

This last is the remainder, so in fact:

$$3\,x^5 + 2\,x^4 - x^3 + 3\,x^2 + 5 = (2\,x^2 + x - 1)\left(\frac{3}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{25}{16}\right) + \left(-\frac{23}{16}x + \frac{105}{16}\right)$$

In general, if the $A(x)$ algebraic expression is divided by a lower order $D(x)$ then we get a $B(x)$ result and $R(x)$ remainder that is: $A(x) = D(x)\,B(x) + R(x)$
Especially interesting is, if $D(x) = (x - a)$ simple first order expression.
Then, $A(x) = (x - a)\,B(x) + r$. So the remainder is a mere number.
But most importantly, if a was a root of $A(x)$ that is $A(a) = 0$, then:

$$A(a) = (a - a)\,B(a) + r = 0\,B(a) + r = r = 0.$$ So in fact, $A(x) = (x - a)\,B(x)$.

So, dividing with an $x - a$ with a being a root, there won't be a remainder.
In short, $A(x)$ becomes a product.
Then, if $B(x)$ has some b root again, then $B(x) = (x - b)\,C(x)$.
Of course, if b is root of $B(x)$ then it is root of $A(x)$ too. So continuing this till the end,

we get: $A(x) = (x - a)(x - b)(x - c) \ldots c_n$.

All these $a, b, c, \ldots$ are roots of $A(x)$ and the last c_n division result is actually the original first co-efficient of A, because this was inherited to $B(x), C(x), \ldots$
This at once proves that an n-th order $A(x)$ can have maximum n roots. Of course, it can have easily less for two reasons.
Firstly, B can have the same a root as A had, so b can be a again, and so on.
Secondly, there was no reason to assume the existence of already a, then b and so on.
We can be stuck at any point. Of course, the reverse is much more obvious.
If $A(x)$ is such product, $(x - a)(x - b) \ldots$ then these $a, b, \ldots$ are all roots.
Simply because one of $x - a$ or $x - b$ or $\ldots$ would give 0 values.
This "root product" form was too beautiful not to be true in general, so it was expected that even if there are no real $a, b, \ldots$ roots, at least formal $u + v\,i$ ones must exist. So:

$$A(x) = (x - a)(x - b) \ldots (x - (u_1 + v_1 i))(x - (u_2 + v_2 i)) \ldots c_n$$

Then with the earlier result that for every $u + v\,i$ formal root, $u - v\,i$ is root too, we can order them so that $u_2 + v_2 i$ is actually $u_1 - v_1 i$. Similarly, putting all of them in pairs:

$$A(x) = (x - a)(x - b) \ldots (x - (u_1 + v_1 i))(x - (u_1 - v_1 i)) \ldots c_n$$

But observe that: $(x - (u + v\,i))\,(x - (u - v\,i)) = x^2 - 2ux + u^2 + v^2$.

So, every $A(x)$ would become a product of the real root factors and second order factors containing the artificial conjugate root pairs.
Some doubted this perfection.
For example, Leibniz claimed that $A(x) = x^4 + 1$ can't be factorized.

Obviously, we can't have real roots, because $x^4 = -1$ is just as impossible as $x^2 = -1$.

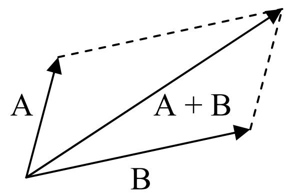
In fact, x would be $= \sqrt[4]{-1} = \pm\sqrt{\pm\sqrt{-1}} = \pm\sqrt{\pm i}$.

So the thing Leibniz doubted is whether the square root of i is itself a $u + v\,i$ combination.
With our modern view we will see at once this combination. But what is this modern view?

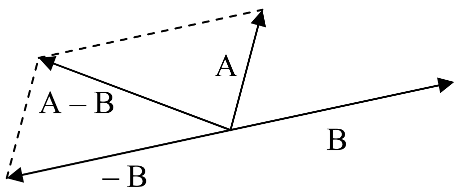
3. Complex Numbers

Amazingly, the Descartes system that was so well known by the times of these puzzles, offered the solution. The reason this wasn't recognized was probably that the old fashioned Descartes vision merely regarded pairs of numbers (x , y) but not sum combinations like u + v i.
Even more amazingly, completely apart from this particular u + v i puzzle that forced math to recognize that the old (x , y) pairs as points of the plane are the real "unreal" numbers, we could have started from those old (x , y) pairs and regard them as sums through a totally different principle, that has nothing to do with $i = \sqrt{-1}$.
This principle is the concept of vectors.
Now this is even more puzzling because vectors were the bread and butter of Newton's mechanics. In fact the modern oversimplified Newton Laws primarily lack the truth due to exactly the de-vectorization. Velocity, acceleration, force are crucially vectors. And Newton saw these vectors as geometrical three dimensional quantities, practically as numbers already.
So calculus too was calculus of vectors right from the start. And still somehow it fell apart into calculus of the three coordinates. It's hard to tell today what was the missing link in their vision to get rid of the old Descartes view with separate coordinates as old numbers above which is space the set of points. Maybe because even sets weren't that conscious yet. After all, Cantor came centuries later. But by today sets did become the household vision in almost a few decades and the old backwardness about vectors still remained.
Elementary and high school level education refuses to regard the coordinates as sub spaces from which the full space is combined by addition. Quite oppositely, tertiary level starts with these as abstraction, so a terrible split exists.
Quite recently with the internet and perfectly exemplified by the Wikipedia math articles, a new tendency emerged. This forces the tertiary abstractions into high school level as the natural road. This is the most dangerous development. A deadly example for the "fascism of democracy".
But returning to the problem of vectors, I think a much bigger reason lies behind this resistance against an algebraization of space.
The continuity paradoxes of the Greek philosophers somehow sunk under the sand of time.
The silver platters of the infinite decimals and the division algorithm to see that fractions are periodical, beautifully solve some of those old paradoxes like the Achilles one.
Even fractals became a colorful resolution rather tan re-awaking of the problems with points.
But the real root of all this is that the effectivity of the naturals became extended to the reals.
To have a value even if it is not calculable exactly is regarded as indeed "real".
So a point is "real" as a point of space but as determinable by measurements it still remains as three distinct and separated coordinate values.
The big question is whether we should attack this seemingly persistent plausibility or not.
I am not sure yet. But we should investigate it definitely.
I did write a short systematic and simple high school level book : Modern Coordinate Geometry.
It will be put on my site and you can decide whether it is convincing enough.
A final comical twist in this whole affair is that the standard tertiary vector view of space uses the letters i , j , k as the unit vectors of the x , y , z coordinates. Thus the old (x , y , z) Descartes point becomes x i + y j + z k . Or in plane (x , y) = x i + y j.
When the u + v i dual combinations became identified also with the point of the plane then also the first became the x and the second the y axis. So actually i became the j vector and nothing just the real part the i vector.
Gauss who discovered the plane meaning of u + v i , was an admirer of Newton and was certain that he hit upon something physical. But no physical quantity corresponded to these numbers that he called complex.
As it turned out later, electromagnetic waves, quantum mechanical quantities and relativity all rely on complex numbers. This makes the previous vector problem even more interesting.

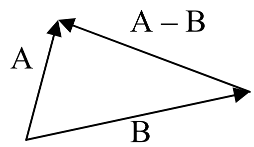
The addition of vectors is common knowledge, usually introduced only in Physics rather than Math. The name “parallelogram rule” tells everything:



Since the negative of a vector is its mirroring in its line, the A – B difference can be introduced as A + – B:



Sometimes they mention it as a new “triangle rule”:



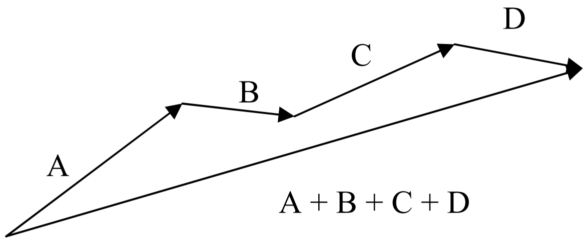
This new version of A – B was not initiated from the same point as A and B, which means that:

Parallel, equal long and same directional vectors must be regarded as same.

This “shiftable” vector meaning is the mathematical. This is also the abstract physical.

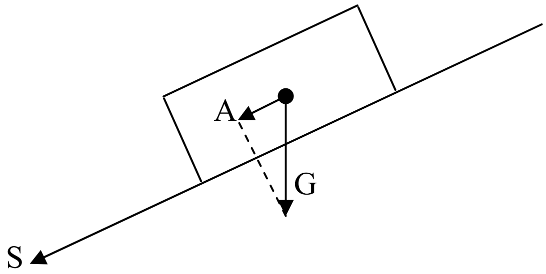
But there are special physical meanings too:
Obviously some forces acting on rigid bodies can not be shifted at all or only in their lines.
So these could be two more restricted physical vector concepts.

The abstract vector addition means that the parallelogram rule can be generalized to a much more heuristic so called “chain rule” that works for many members:



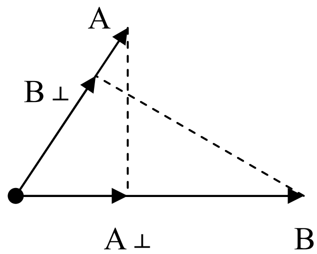
The real business of vectors starts with introducing multiplication.
There are three different kinds.
The simplest relates to the physical meaning of work and thus energy.

Work happens when a force moves. So, static that is standing forces don't do work.
That's why a book can sit on a table for ever without gravity doing any work.
If the book falls, gravity will do work namely increases the speed of the book.
This work or invested energy can be recovered because the book will have a kinetic energy.
If the book slides down on a slope, its speed will increase less because the invested energy will be partially lost on the friction. The book and the slope will gain some heat.
If the slope is perfectly smooth that is frictionless than the end speed of the book will be the same. The slope is still longer than the straight fall so we might think there is a contradiction because the same gravity force, the weight of the book was acting on a longer length so the work should have been bigger. The resolution is simple. Since the G gravity points down, not in the direction of the S slope, only the projected length of this force onto the slope line acts as actual force A:



Observe too that a table is itself a slope having no declination at all. The projection of gravity here is obviously 0 because it is perpendicular. Since the acting force is 0 it doesn't move.
So this first multiplication calculates a product G S as merely G A using a projection first.

The surprising fact is that this multiplication is commutative which means that its exchangeable in the order of its two members. So it doesn't make any difference which one of the two vectors is the projected. To see this is easy because it comes out of the similarity of two right angle triangles with the common angles between the two vectors:



$$\frac{A\perp}{A} = \frac{B\perp}{B} \quad \rightarrow \quad A\perp B = B\perp A$$

As opposed to this symmetry, the real application for work requires an asymmetrical extension of this multiplication.
In quite general, it can be called as the "partitioning of the product".
The two members we multiply are not fix they are changing in time or space.
Usually already this hides an asymmetry, so one can be regarded as function of the other.

In our case for example both the slope and the gravitational force can change in space.
But since the slope is a lower dimensional curve or surface and the body must move on it, we regard this S surface vector as the independent and the G gravitational force as its function.

So, we partition S as $S_1 + S_2 + \ldots + S_n$ and regard in these small sections as fix vectors, that is as tiny little slopes.
On contrary, the other vector and product member G is not partitioned.
It is allowed to change continually.

Of course we couldn't use infinite many different values for that either, so actually we do something similar as partitioning but it is not requiring dissecting and thus reducing the length rather simply a random choice of fix $G_1, G_2, \ldots, G_n$ values for each partitioned S piece. Namely, each G_i is chosen from values that appear on S_i.

Then $G_1 S_1 + G_2 S_2 + \ldots + G_n S_n$ is a good approximation of $G S$.

Indeed, if for example all the G_i values are taken as same G then :

$$GS_1 + GS_2 + \ldots + GS_n = G\,(S_1 + S_2 + \ldots + S_n) = G\,S$$

The S_i members can also be symbolized by dS, meaning differences of S values.

Also, the summation can be abbreviated by $\sum$ and so $\sum G\,dS$ is the approximation.

The perfect physical meaning that is the actual work comes about by allowing finer and finer partitioning of the product and taking the limit value of these $\sum G\,dS$ approximations.
The symbol for that limit is the "curvy" version of $\sum$, the integral sign.

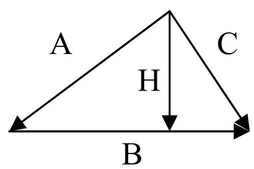
So the work is $\int GdS$.

Observe that each little product piece that we added must involve tiny little projections of G to the dS slope piece.

The second multiplication of vectors is the opposite of the previous in the sense how we regarded the optimal actions by directions. Above the parallel was the optimal with full length multiplication and perpendicular was the minimal, 0 effect. Now we go oppositely.
But is there reality behind this?
Well, Archimedes said: "Give me a fix point and I move the world."
He was referring to use a long enough lever which can multiply our force up to arbitrary large.
What he didn't mention is that using a force to a lever still must be applied with a certain common sense. Namely, you don't push the lever in its own direction towards its axel, rather perpendicularly. In fact, if you push it in an other angle, you'll loose some force.
So, the lever arm as vector L multiplied by our force F gives the action exactly by such opposite multiplication which is called vectorial product: $F \times L$

But an even simpler example comes out by sheer geometry.
Indeed, perpendicular multiplication is the area.

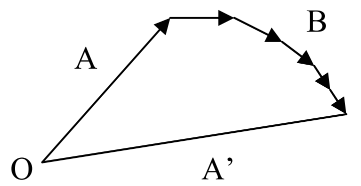
In fact, the perpendicular projections of either A, C sides to a B base line is exactly the H height of a triangle:



$$H = A^{\perp} = C^{\perp}$$

So this vector product $B\,A^{\perp} = B\,C^{\perp} = B\,H$ gives the area.

Partitioning B and allowing A to change continually, this partitioned product and finally its integration can give the actual area of a “path triangle”.
That is of a continuous curve used as “base line” with an O fix point as “corner” across:



This was crucial in Kepler’s Second Law but as it turned out it has nothing to do with the planets.
Every central force moves an object so that this swept area is the same under same times.
It simply means the exact rule how a central force speeds up the object as it comes closer.
So spinning a rock on a rope and pulling it in carefully without jerks will obey this same law.
Every second, the rock will sweep the same area. Out there with long rope slowly, while when the rope is short it must go fast to sweep the same area.

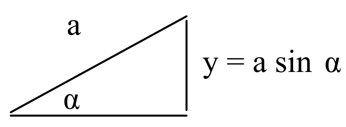
Both of these physical vector multiplications were already known when Gauss discovered the third new one that rules these $u + v\,i$ dual forms. Of course we might say that the rule is already there because we can multiply any two such expressions member by member, obtaining four members and then separate the real and imaginary parts. But this is more like a blind calculating method, a meaning for computers, not thinking humans.

Strangely, the true meaning still has a less grand and more mechanical version too. So now that we started already with totally blind one, I should reveal this first. This has to do with an actually more basic departure from the Descartes system as locating points. Indeed, why should we use two perpendicular coordinates at all. We could use only one with an origin on it and from there we measure an α angle. Knowing this direction, then we can give the a distance of the point from the origin. They together as $< \alpha , a >$ are the polar coordinates.
It is used to prove Kepler’s Law from Newton’s Laws and is used in navigations today.
The amazing fact is that if we change $x + y\,i$ forms as (x , y) Descartes coordinates into polar $< \alpha , a >$, then the multiplication becomes a simple multiplication of the lengths and addition of the angles.

So, if $x + y\,i = (x , y) = < \alpha , a >$ and $u + v\,i = (u , v) = <\beta , b>$ then

$(x + y\,i)(u + v\,i) = <\alpha+\beta , a\,b >$

Quite easily, the Descartes x , y coordinates can be obtained from the polar by the good old trigonometric functions cosine and sine:

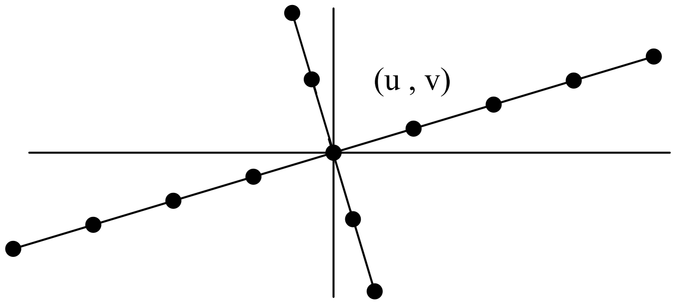


So, we have a treasure chest of deriving trigonometric formulas by the above multiplication rule.

Bits and pieces of all these were recognized way before Gauss and yet a full picture was missed.

Now we come to the grand abstract meaning of multiplication which wasn’t explicitly emphasized even by Gauss.
Strangely, in this meaning, the obvious commutativeness of the previous is lost. Indeed, above both the $\alpha+\beta$ and $a\,b$ parts of the product are commutative, so the vector product is too.

In this meaning, one of the members is regarded as a new unit, initiating a whole new Descartes system inside the original.
This is achieved by regarding this chosen one, say $u + v\,i = (u , v)$ as new unit of a new x axis. This of course determines the perpendicular new y axis with same unit length too:

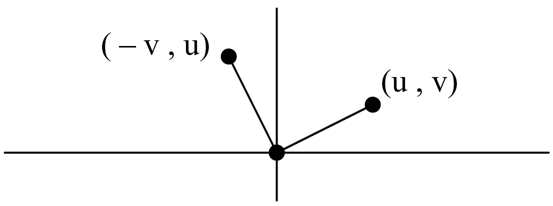


Now we claim that $(x + y\,i)(u + v\,i)$ is simply placing $x + y\,i$ into this new coordinate system. That is, using x in $u + v\,i$ direction with this as unit, so finding $x\,(u + v\,i)$ as point or rather vector.
Then using y similarly for the perpendicular 90^o turned version of $u + v\,i$.

Adding these together must give the product so: $(x + y\,i)(u + v\,i) = x\,(u + v\,i) + y\,(u + v\,i)_{90}$

The real question is of course what this $(u + v\,i)_{90}$ is.

It is quite simple: $(u + v\,i)_{90} = (-v + u\,i)$:



Most surprisingly, our original mechanical multiplication gave this very fact but we didn't see its meaning:

$(x + y\,i)(u + v\,i) = x\,(u + v\,i) + y\,i\,(u + v\,i) = x\,(u + v\,i) + y\,(u\,i + v\,i\,i)$

and $(u\,i + v\,i\,i) = (-v + u\,i)$

Our grand picture gives the polar multiplication too. Indeed, placing $x + y\,i$ into the subsystem with $u + v\,i$ unit, clearly means the angle measured from $u + v\,i$ and the length of $x + y\,i$ is also multiplied by the length of $u + v\,i$ since this is the unit length.

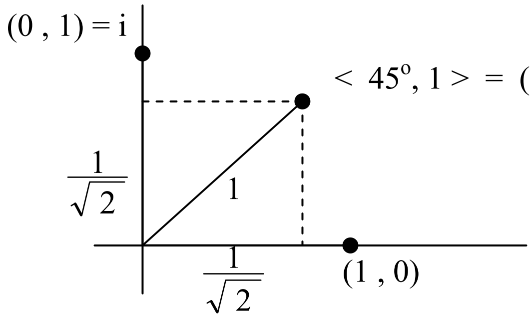
Finally, we can see what a conjugate pair of complex numbers mean.

They are symmetrical to the x axis, so they have α and $-\alpha$ polar angles.

Thus their product has 0 angle so it is on the x axis, so it is a real.

But such 0 angle and thus real product could come out from merely symmetrically angled but not equal long two vectors too. So why did we only get perfectly symmetrical conjugates?
Because not only the p product but the $-s$ sum has to be real too.
The parallelogram rule then shows that they have to be equal long too.

Now we are ready to see the complex numbers that reveal what Leibniz claimed as impossible.



$$< 45^o, 1 > \; = \; \left(\frac{1}{\sqrt{2}} \, , \, \frac{1}{\sqrt{2}} \right) \; = \; \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \; = \; \sqrt{i}$$

The picture tells everything: We placed a point on the corner of a square with 1 diagonal.

Its polar form is trivial and its Descartes coordinates are obvious from the Pythagoras theorem.

Or we can see that the squares area is half, by envisioning a square on the diagonal itself.

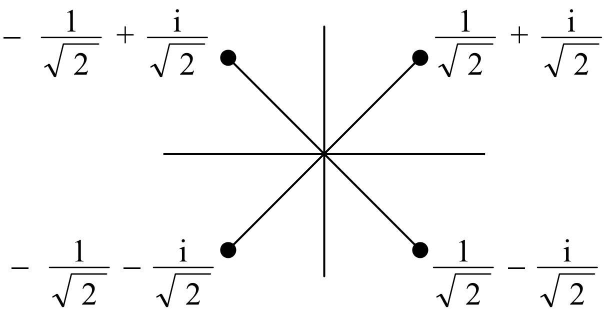
On the picture we claimed that this point is $\sqrt{i}$.

Indeed, its polar form is $< 45^o, 1 >$, so its square is $< 90^o, 1 > \; = \; (0 , 1) \; = \; 0 + i \; = \; i$

But we can go the "hard way" too, from the complex form:

$$\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^2 \; = \; \frac{1}{2} + \frac{i^2}{2} + 2\, \frac{1}{\sqrt{2}} \, \frac{i}{\sqrt{2}} \; = \; \frac{1}{2} - \frac{1}{2} + i \; = \; i$$

The negative of this vector must have the same square. This is our point mirrored to the origin. The other two solutions of $\sqrt[4]{-1}$ are the conjugates of these two, so the four solutions are:



So $x^4 + 1 =$

$$\underbrace{\left(x - \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right) \left(x - \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right)}_{(x^2 + \sqrt{2}\, x + 1)} \underbrace{\left(x - \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right) \left(x - \left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right)}_{(x^2 - \sqrt{2}\, x + 1)}$$

Euler observed this, thus correcting Leibniz and putting the faith back firmly, that all A (x) can be factorized.

Unfortunately, he couldn't prove the crucial existence of a complex root for all $A(x)$.

He was especially obsessed with this total factorization, because he realized that it at once proves the generalization of the rule we used at the second order formula for the sum and product of roots:
For arbitrary $A(x)$, again we have to divide with the c_n highest co-efficient, that is avoid it.

So if $A(x) = x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$ then, $-c_{n-1}$ is the sum of the roots.

But now the product of them is $\pm c_0$ depending on n being even or odd.

Indeed, using just $a, b, c, \ldots, r$ as roots regardless real or complex, we have:

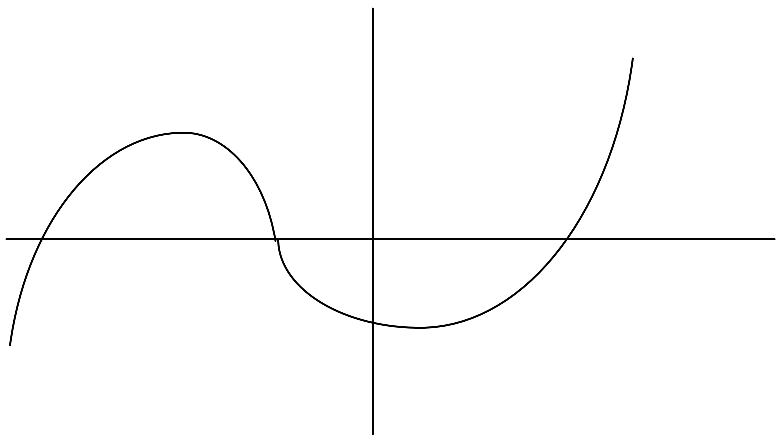
$A(x) = (x - a)(x - b) \ldots (x - r)$

Using all x we get x^n. Using $n - 1$ many x and one root, we get all of them with minus, so

$-x^{n-1}(a + b + \ldots + r)$. Finally:

Using all the minus roots, we get their product, but the sign is negative if we use odd many.

The worst thing about the missing evidence for the existence of real or complex roots was that for odd order $A(x)$ it was already trivial that real root exists.
For a second, we might say it's okay, because then at least the previous rule for example, is still true for odd order $A(x)$. Not so!
Lets remember that the factorization was done step by step, always assuming a root.
So we have to go through even ones, even if the original is odd.
At any rate, lets see why a real root is obvious for odd n.
Simply because these $A(x)$ have arbitrary big negative and positive values:



Indeed, let $A(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$ and c_n be positive.
If x goes towards $-\infty$ then $c_n x^n$ dominates all the other members as a huge negative value, since n is odd. When x goes towards $+\infty$ then of course, the value becomes $+\infty$.
If c_n is negative, then the picture is opposite, going from $+\infty$ to $-\infty$.
In either case, the function has to cross the x axis.
With even n, the $c_n x^n$ is again dominating, but it becomes the same infinity, when x approaches either $-\infty$ or $+\infty$. So, a crossing is not guaranteed.
How can such stupid little difference cause such a deep problem?
No wonder, all mathematicians went nuts about this whole affair.

4. Proving the Fundamental Theorem of Algebra

Even when finally Gauss as a young man saw the light in the tunnel, namely that the u + v i combinations are actually the points of a plane as new numbers, he still kept the old distinction of real and non real roots. Most importantly kept the coefficients real.
He didn't transform the whole question into the plane.
That would have been going from

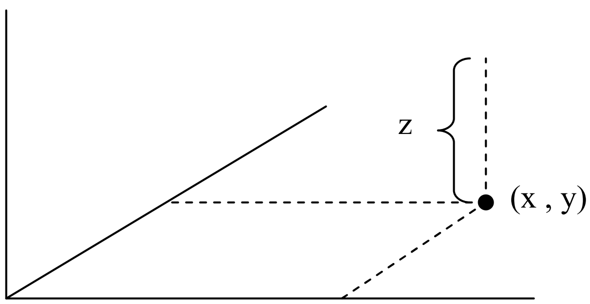
$$A(x) = c_n\, x^n + c_{n-1}\, x^{n-1} + \ldots + c_1 x + c_0 \qquad \text{to}$$

$$\mathcal{A}(P) = C_n\, P^n + C_{n-1}\, P^{n-1} + \ldots + C_1 P + C_0 \,.$$

This is perfectly meaningful with $P^n = P\ P\ \ldots\ P$
So it means multiplying P's angle by n and taking the n-th power of its length.
The C_n coefficients are now points themselves and their application is clear too as turnings and stretchings again.
Adding all these turnings and stretchings together, we get the $\mathcal{A}(P)$ value.
The odd - even distinction of real algebraic expressions or parabolas disappears, but the major fact that $C_n P^n$ dominates the smaller ones remains. For example if P is big enough then:
Even if C_{n-1} is much bigger than C_n, the $C_n P^n$ will be bigger than $C_{n-1}\ P^{n-1}$.
Here of course, "big" is meant by the point's distance from the origin.

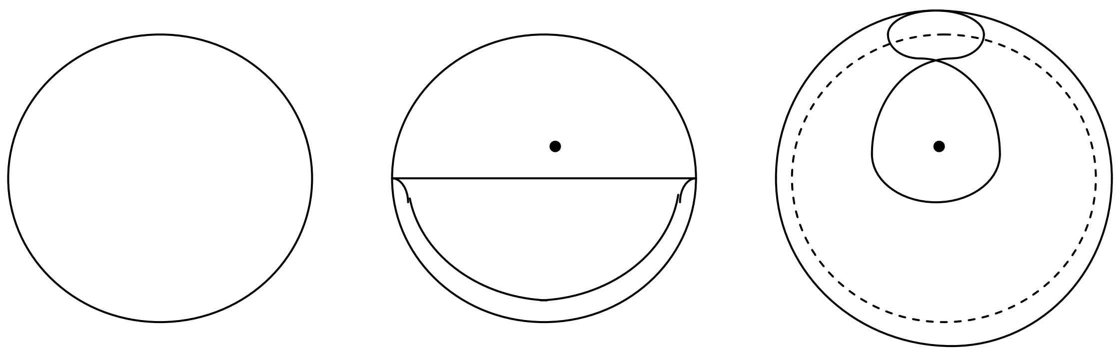
While earlier the plane was a subjective tool where mathematicians drew the graph of the real $A(x)$ as $y = A(x)$, now the x , y plane is itself the domain.
This loss of convenience was a strong reason not to realize the transition.
Today even more, the Descartes system as a silver platter is so strong in high school and even in every day life as the field of graphs and charts, that it covers up the realities of functions.
In high school we never even mention the fact that the $y = x^2$ parabola is an illusion, because we are dealing with a function that orders numbers to numbers, in one single line.

If we use the plane as a domain, then the $Q = \mathcal{A}(P)$ creation of Q points from P, couldn't be graphed in a similar way as the Descartes system, even in space.
Indeed, three dimension can only order single z values to (x , y) points.

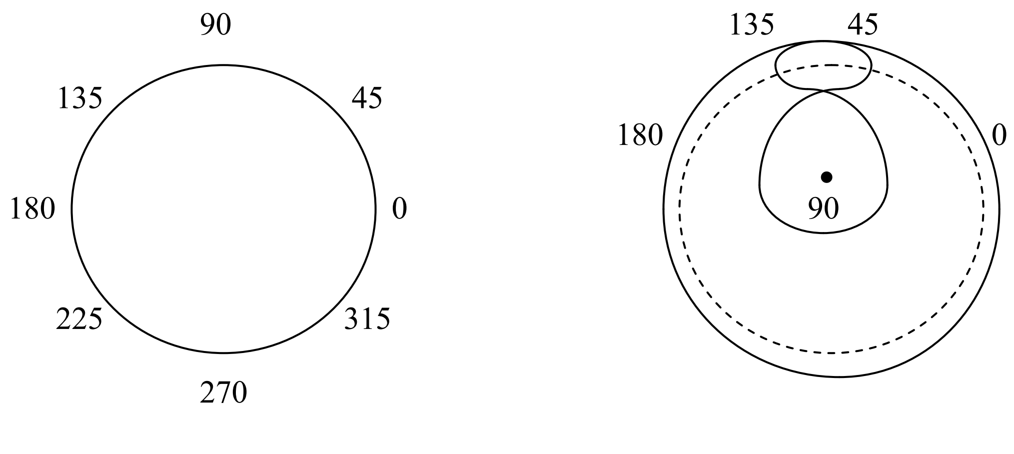


To "see" $Q = \mathcal{A}(P)$, we would need four dimensions. But this is the stupidest complaint!
Giving up the graph convenience is a step towards truth!
A truth forced upon us by the three dimensionality of our space.
I'm not saying that we should give up Descartes charts on economic news bulletins, but I do claim that when the domain is the plane, then it is didactically ripe to go back to reality and visualize the $Q = \mathcal{A}(P)$ orderings in the single plane.
This will involve time as the new false illusion in a strange two-leveled way.

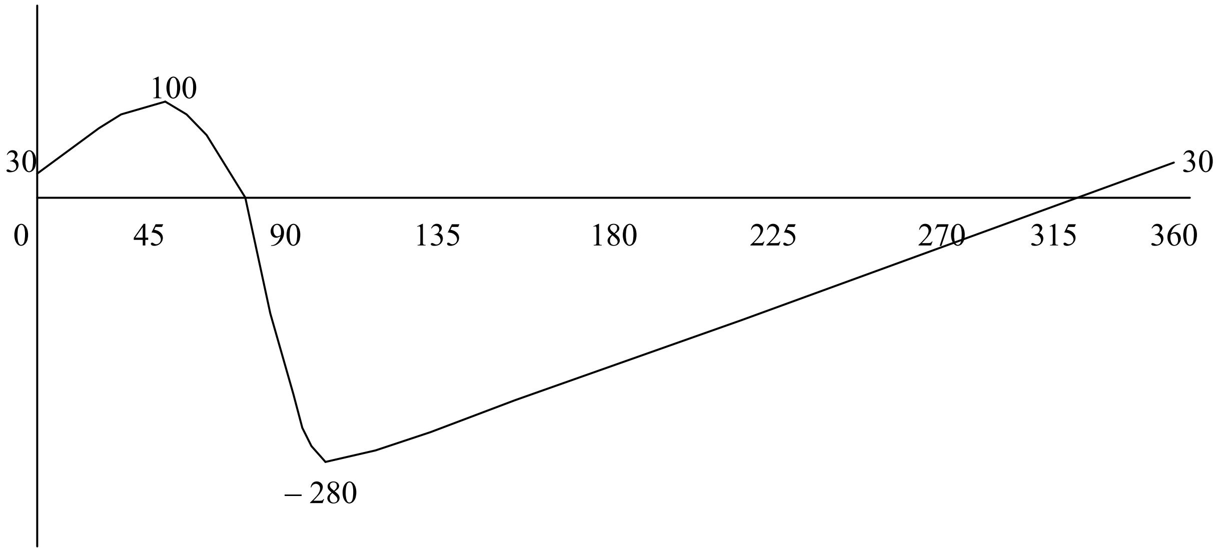
The continuity of $\mathcal{A}$ (P) means first of all that moving P to new points, the Q value moves too. So here time is only potential. The usual "spaced" version of this vision is that the set of P values, the domain is the infinite plane as a rubber sheet and the Q values are a bent, wrinkled, stretched version. So we don't visualize how the points of the rubber sheet moved, only the initial flat and final transformed version is important.
This already helps a lot to see the major problem.
If for example, we have a finite round table and the rubber sheet is glued on the edge, then no matter how we deform it inside the table, it will always cover.
If the edge is not glued, then of course, we can fold it over and leave half or even more of the table uncovered. We can make any point inside the table become uncovered.
Then, we can even stretch the half doubled edge part of the rubber sheet to cover the whole edge of the table, but still leave any uncovered point on the table to stay uncovered.
If however right from the start, we are only allowed to move the edge of the rubber, on the edge of the table around, without leaving it, then again, the covering must remain.
Now if the "history" of the rubber sheet doesn't count, then what's the difference between the folding and re-edging or a strictly edge moving? One allows a hole, the other not.
In addition to this problem, we have the infinity of the plane, so strictly speaking, keeping the edge in itself, is impossible.
Is the dominance of $C_n P^n$ enough to replace the idea of the edge moving in itself?
Clearly not. It only corresponds to the edge ending up on the edge!
We can easily find infinite transformation that keeps the far away points far away and still will leave a hole in the values. Easiest to achieve this, is by creating a hole in a unit circle, as on a table, and then blow out the whole transformation to infinity.
So why are the algebraic $\mathcal{A}$ (P) transformations strictly inside edge motions and never folded ones, allowing holes. This is the real question, not why do we have roots.
If $\mathcal{A}$ (P) picks up all points, it picks up the (0 , 0) origin too, so we have root obviously.
The explanations are already hidden in the finite table situation!
It is not true that the edge folded motions end up the same as the moves with the edge in itself!
This picture shows such folding and re-covering of the full edge, with a dot left uncovered:



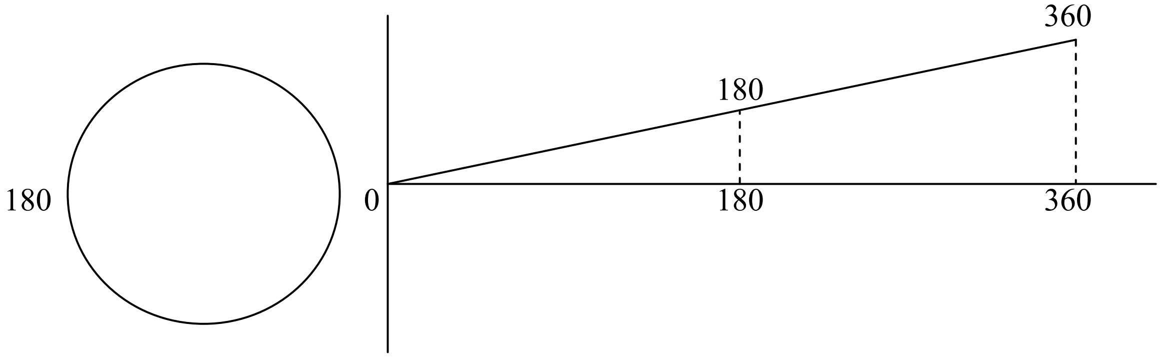
Looking from such left out points, our effort to cover the edge looks quite different if we now introduce a new time aspect, namely the "looking around".
In fact, this can be seen even better if the original sheet had angle numbers around it.



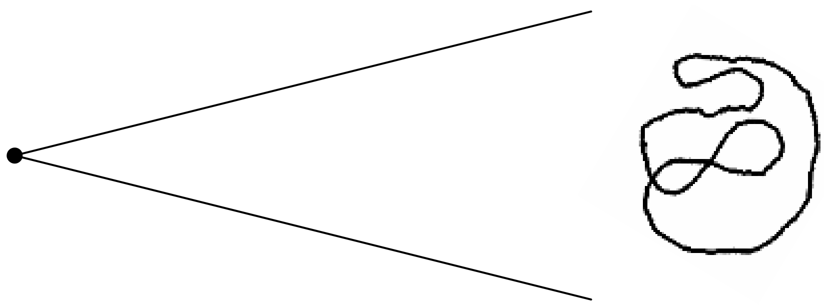
If such left out observer follows the numbers in this “cover up”, then a strange double circling is happening in his actual angles with two reversals in the direction.
This can be seen even better by graphing the angles of view as a function of the original angles.



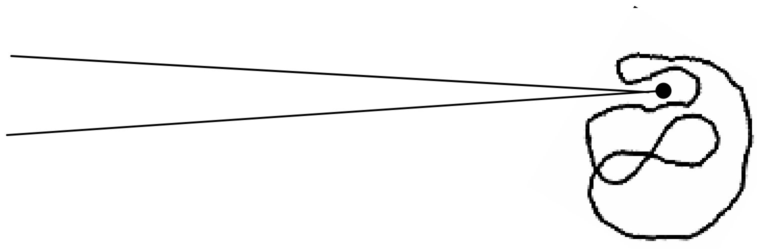
It doesn’t look that strange, but lets compare it with a real full circle without any distortion:



It’s not the proportions that are important, in fact, the view angle was in much smaller scale than the original angles. The vital fact is that here, the view started from 0 but ended on 360^o.
These two angles are the same in the plane as measurements, but not as the track of the journey.
Above, the view angle actually returned to the 30^o . So in fact, we didn’t do a circle at all!
Most amazingly, actual circling of a point can be even more complicated than these two.
We can go around more than 360^o and still return to accomplish nothing! On the other hand, we can also circle round and round with more than one full 360^o above or under the initial angle. In short, the view angle can end up to be $\pm\ m360^o$ to the initial.
If we are far away from a loop then we can be sure that we are outside of it, because it fits into our view with a much less than 360^o angle. So it doesn’t surround us, even subjectively.



A less trivial case is when the “escape” angle is small but still guarantees outsideness:

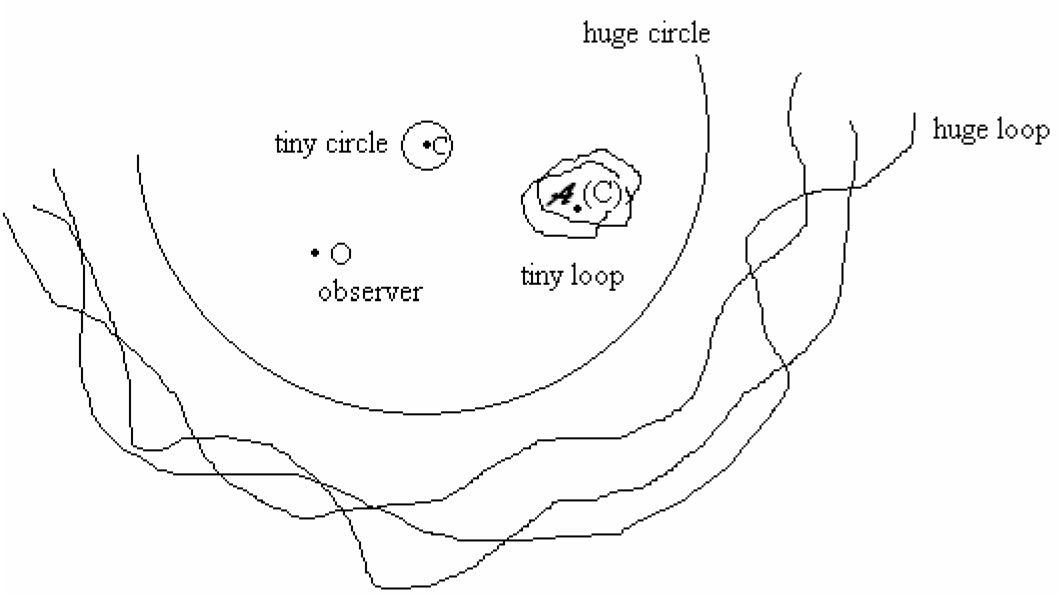


These escape angles at once guarantee that the loop angle can only be 0.
But 0 loop angle in general, is much harder to distinguish from $m360^o$ visually:

From the five points in this picture:
One is obviously outside by an escape angle.
One has 0 loop angle without escape angle.
And three have $m360^o$ loop angle.

The crucial fact is the following:
If an observer is fix, and a loop is continually moving but never crosses the observer, then:
The loop angle can not change. In particular, if again, the loop is not crossing the observer:
Trivial outsideness with escape angle can only change into 0 loop angle:

Now the domination of the $C_n P^n$ member can show its additional feature!
It is not only becoming further than any fix observer as P increases, but as P goes around once, $C_n P^n$ goes around n times. So, if P increases in circles, its $\mathcal{A}$ (P) images will create $n360^o$, that is non 0 loop angles. That's why hole can not appear.
The precise argument goes as follows:
Lets choose an arbitrary O observation point. We will show that it has to be an $\mathcal{A}$ value.
Lets choose a second C point. If $\mathcal{A}$ (C) is O, we are finished. If not then there is a small enough circle around C, so that its image by $\mathcal{A}$ is such a tiny loop, that O is outside of it obviously, that is with an escape angle.
Now increase the tiny circle continually! Its image will eventually increase too.
In fact, after a while, the circle is big enough that its loop image by $\mathcal{A}$, will definitely surround O, namely with $n360^o$ total loop angle simply by $C_n P^n$ dominating :



But a continually changing, loop with 0 loop angle cannot become $n360^{o}$, unless it passes through the observation point O.
The loop points are of course all values of $\mathcal{A}$ and so O must be too.

History will not reveal how much of this was clear to Gauss because he decided not to reveal it. He did return to newer and newer proofs for this so called Fundamental Theorem of Algebra. But he was chasing more an illusion of exactness, than a better picture for others.

5. The Vultures of Formalism

Gauss, not stripping away completely the $\sqrt{-1}$ form and reveal the geometrical meaning behind what he called the $x + y\, i$ complex numbers as merely (x , y) points, was a didactical laziness. He simply didn't care about over-analyzing the vision.
He was onto something much bigger. He was chasing a new truth. He was talking about his new numbers as a "new layer in the world of shadows". He was physical.
His idol, Newton, was something very special in human history. This is beyond being one of the greatest minds and yet one of the most disturbed geniuses at the same time.
The scientific uniqueness of Newton is that he was the first and last mathematician physicist.
Conventional science historians would again try and disprove me by bringing up formal counter examples about other scientists crossing over. But the simple truth is that being a mathematician is a category and being a physicist is another. Understanding as opposed to creation is of course open for both fields, in fact for everybody.
But the truth is even more shocking! Before Newton, there were no physicists.
Researching reality, going from alchemy to chemistry, measuring water and air pressure are all very nice things, but have nothing to do with the mind frame that a Maxwell, Einstein, Heisenberg possessed. The crucial shift of course is most detectable looking right before Newton. Two minds were at the verge of crossing over. Galileo, this wandering, sweet and stubborn rebel, and Kepler, a lost mathematician. If they had paid more attention to each other, they could have crossed over. Usually, when Newton's revolution is explained, we go into how forces, gravitation especially combine the earthly and heavenly worlds of Galileo and Kepler.
But this slides over something much deeper. A hidden new role of mathematics.
Kepler was the first to inject math into nature. Before him, the math of physics was too simple to notice it. He, using Tycho Brahe's data collection could only hit upon the idea of ellipses, because he was a closet mathematician. Then claiming that God put the planets on ellipses, was where he missed to be a physicist.
Newton deriving the Kepler laws from the laws of forces, made the first grand offering on a new altar. Newton re-defined God. After Newton, anybody believing in white thrones or lotus flowers alone as the realities of the other side, is bound to be an ignorant and false witness.
Math does not have to be injected into nature! Math is behind reality or math is the deeper reality! To play with this deeper reality is a freeing of one's spirit. And indeed, many became mathematicians throughout the ages. But to become a physicist, is a whole new "game". It is much younger than becoming a mathematician, and so "we" don't have a correct picture of it yet. And here by "we", I meant philosophers, the only ones who care to see the biggest picture.
So, all these complicated thoughts come back to the explanation, why Gauss didn't care about explaining and clarifying the visions. He was a secret, closet physicist. Instead of feeling the immediate solution to his pain, he gazed into the future. Where complex numbers are used in Quantum Mechanics. He saw something far away. That's the usual cause of not seeing what's right in front of our eyes. But now I want to talk about the vultures.
This is the real reason why Gauss' and all other geniuses fault in not caring enough to explain can have devastating effect on history. The result is the army of epigones who teach the teachers not to teach, create a cesspool of rotten food for the minds.
Since mathematics is playing in the garden of truth, it is easy to stray in it. To require math as a separate subject is actually the trick of the devil, to lock people out from its paradise.

Physics, which is a much newer mystery, can not be abused as much as math.
So it is simply locked out from the education itself. In physics, the lies can not be piled up with the same formal derivations as requisites. So the fact that people don't possess the minimal vision of physical reality is left unquestioned. It just melts into the technological second nature that we live with but live without in spirit.
Very few mathematicians come to this strange, final analysis. In fact, I don't know anyone besides me, who is a champion of New Math and at the same time, knows that math shouldn't be a subject on its own. Only integrated science is correct didactically.
The morsels of common sense must be guided to abstractions. Abstractions cannot be taught on their own. To teach all kinds of complicated math to kids, who can't even visualize the atmospheric pressure, is an obscenity. But most sadly it is an intentional conspiracy of the devil. So we arrived from Gauss not caring, to a world misleading its children. Once somebody loses sight of the big picture, then the veil of spell will lead him to his particular truth, which can still point to the right directions. This article is no different! It approaches something very simple, in a very round about way. Falling and climbing again. Up until now, it seemed about math.
But accepting math, even as a subject is a lie, as I just said. I already mentioned about the changing methods in the teaching of second order equations, and how ridiculous is to solve it by guessing roots from the coefficients. Other whole fields, like calculus is going through cyclic re-evaluation by the narrow minded vultures. Calculus in, calculus out. Every ten years, they re-play this game. As slowly the old dogmas of trigonometry fade, and new "hip" fields like probability, are introduced with the same dogmatic approach, the teachers are trained to obey, so the students have no chance in hell. The stupidity of math on its own, is almost at the point of revealing itself. Forty years ago, it was argued whether a common high school curriculum is meaningful at all. They said, "Lets do some more real education towards jobs, not just for potential tertiary levels.". Against this I could say that schools should be windows to the universe, not doors to the world. But that's beside the point now, when 90% of kids who diligently learn math, are doing it just to get into medical school and never use it anyway.
The rottenness of education is beyond repair, nobody believes that school can make you better. It is the test to be bad enough for the world.
There was a time when it already become clear why education is a lie.
Timothy Leary said, "Turn on, tune in, drop out.". Without going into the depth and clarity of this, just observe that "drop out" was the end of it. No didactical analysis was required to see that. But then, for a short time, it seemed that instead of the "knowledge is power" the "seeing is freedom" can be true. The fact that this polarizing of the attitudes could die off, and become an "era", a potential "been there, done that", already shows that the trouble is much deeper than anybody can guess. The taste of the word "establishment" still remains, so hundreds of years from now, Timothy Leary will receive his place in the philosophical hierarchy, but the truth of the sixties is blocked by the beast it tried to unlock. No knockouts in this fight, not even scores. The mediocre nimrods who saw nothing more than unwashed rebels didn't win. The Nixons, the Reagens, the Bushes, the conservative stupidity of "honest day's work", cannot even put a finger on the pulse. Their own doesn't exist. Zombies or skeletons, as Ginsberg called them. Their focusing is self induced narrowing of the iris! Anything against the establishment painted as dangers against freedom. The puppets of establishments are hypnotized, but there is no conspiracy, there are no puppeteers. The Beast is bigger than Man and it doesn't need men to enforce its will. The truth of the lies must come out in even more tangible ways, undeniable, irrefutable. Common sense nonsense is the goal. When all details of the emperor's clothes are adored, can it only become transparent. So it's all good. The ex-hippies are surfing the net, and Wikipedia can become part of the school system. The stupidity is the same and the illusion of freedom is stronger. Before, the libraries were the overwhelming collections of useless details. Now they are the sane refuge from the bewildering emptiness of the internet.
Where will this article end up? I don't know it yet.