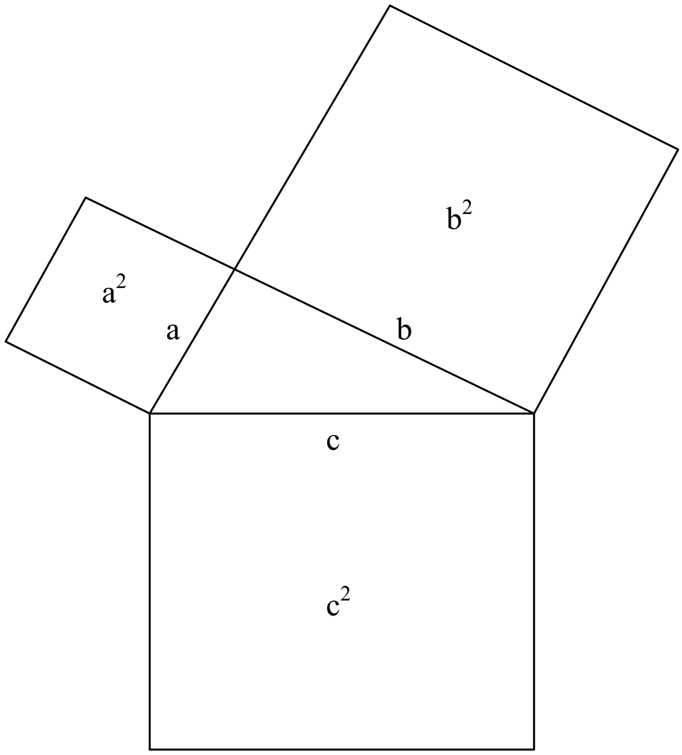
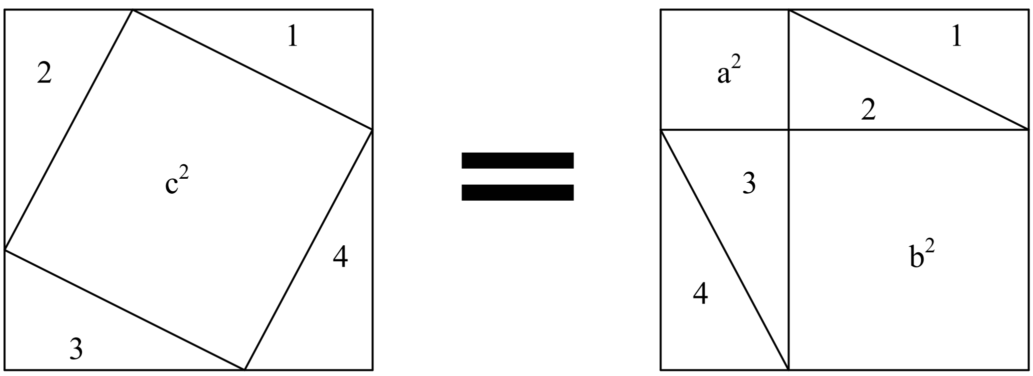
Generalized Pythagoras Theorem

The Pythagoras theorem came from India through Arab mathematicians to the Greeks. It claims that if we draw squares on the sides of a right angle triangle, then the two smaller ones together equal the biggest:



The Hindu proof for it applies a trick of adding four copies of the triangle to both $a^2 + b^2$ and c^2, obtaining in both cases an $a + b$ sided big square:



Thus of course, both $a^2 + b^2$ and c^2 are $(a + b)^2$ minus the four triangles.
This Hindu method is in sharp contrast to the Euclidean or axiomatic process.

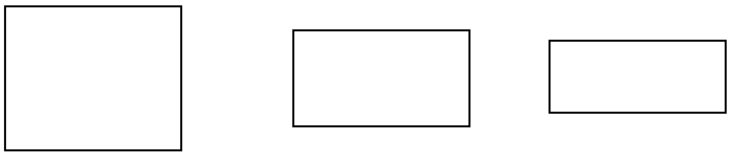
Indeed, in a Hindu proof, one could use anything that is intuitively obvious, without spelling it out. In fact, speech or writing should be minimal and the listener would just “nod” to accept the argument from the pictures. If on the other hand, we strictly list all the accepted assumptions or axioms, then the proof would be a chain of logical steps, using only the axioms or already proved facts. Writing this chain down as a derivation, we wouldn’t even need pictures. When this method reached its peak in the 19th century, then it became clear that a formal derivation without its meaning content can even be applied to alternative meanings. The non Euclidean geometries induced this new final stage of formalism, that lead to the discovery of Mathematical Logic. Together with Cantor’s Set Theory, it provided the foundation of the whole mathematics. The contradiction that all this has nothing to do with how actually mathematics is discovered was obvious, but seemed too hard a problem to attack. Artistic creation is regarded as kind of a mystery anyway, so why should scientific creation be different. A seemingly less interesting problem is that the axiomatic method is not only alien to the creation, but also to the understanding of math. Teaching well or didactically was an issue for some scientists for a while, but by today the insane Formalism completely took over the education system in all levels. The obsession with definitions and proofs without showing the bigger pictures is a menace that can not be changed in its own, because it relates to the wider corruptedness of society.

It’s important to see that going back to the pre-Euclidean Hindu method is not a solution. That’s what the pre-Formalist, honest science educators tried to do. But the Hindu proof of the Pythagoras theorem, though interesting, still relies on the arbitrariness of the created sequence of intuitive facts. Only a modern anti-Formalist approach can be useful today. Of course, the science books of the forties and fifties are still better than the new Formalist insanity, but they can’t be continued directly. A new militant anti-Formalism must be achieved. This is what our site is about. There is no system yet, we have to show how one should act, case by case. So the following generalization of the Pythagoras theorem is such a concrete example of didactical clarity. Full picture given, with full simplicity.

Roughly speaking, this generalization is huge, because it applies to non right angle triangles, and also we’ll talk about rectangles, instead of squares. In fact, the heart of the whole wider picture is the comparison of rectangles.

The two main special relations between two rectangles is them being either “similar” or “equal”. Similar means having the same ratio of the sides. This is exactly what movie or TV screen aspect ratios refer to:

Full Wide Letterbox

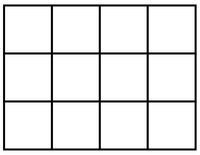


1.3 1.85 2.35

These numbers refer to the ratios of the other side regarded as 1.

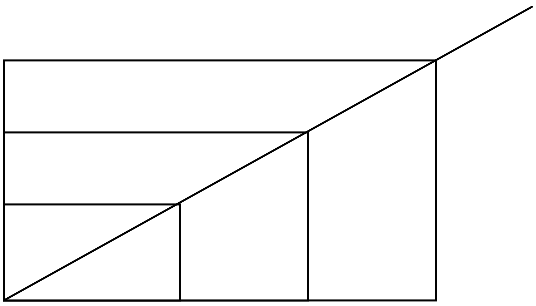
They can be of any size, so actually the ratios are the a : b ratios of the sides.

The other relation of rectangles “equality” refers to the area, which is not the ratio of the sides, rather their product:

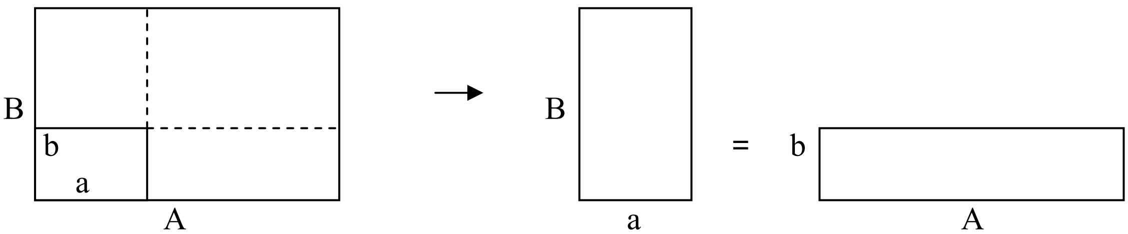


area = 4 x 3 = 12

The fundamental law of the similar rectangles is that placing them in a same corner, the opposite corners all lie on a single line, they are “linear”:



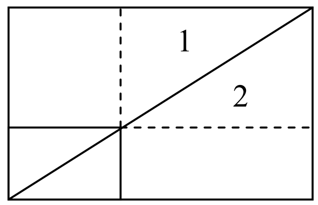
The second simple but interesting fact is that for two such similar rectangles, the smaller, when continued will cut equal parts from the bigger:



The reason for this equality can be seen two ways. Algebraically, the small rectangle’s a , b sides have the same ratio as the big’s A , B that is: $\frac{a}{b} = \frac{A}{B}$

Multiplying “over”, that is multiplying both sides with b B , we’ll have: a B = A b

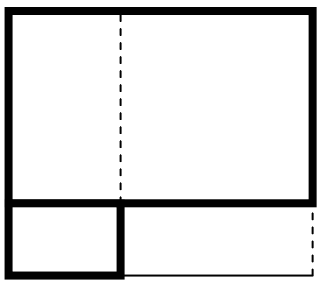
A more interesting way to show the equality is to rely on the linear feature:



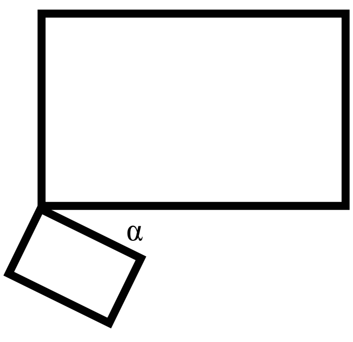
The 1 and 2 triangles are equal, just as the two big ones. Then, the a B rectangle is this big triangle minus 1, while the A b is the big minus 2.

This similarity created equality can be generalized.

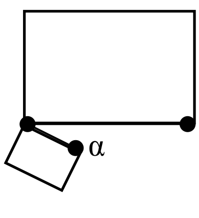
First of all, an alternative, but more symmetrical way of saying the previous claim can be, if we move the small rectangle down outside the big:



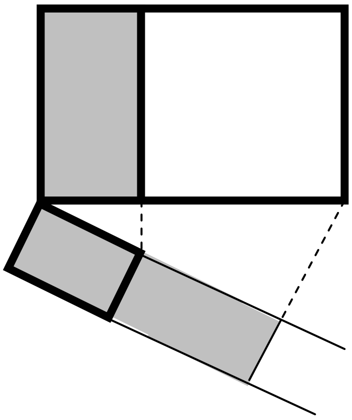
Indeed, now the two equal pieces are “cut off” or “extensions” of the two rectangles, by the other. Now we’ll allow to even turn them around the common corner and thus, having an α angle between them:



The cutting off or extension now becomes a bit ambiguous. We could just continue the sides till they cross the line of the other rectangle, but then we would have to change those lines to get parallel with the other sides. A simpler way is not using the sides, rather just the two end points of the α angled sides:



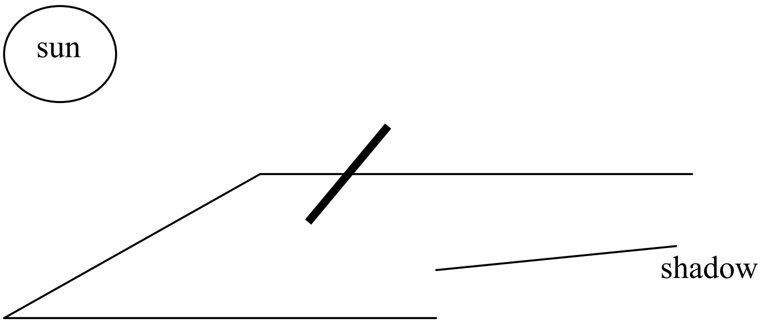
From these two points, we can use single lines going parallel to the other sides:



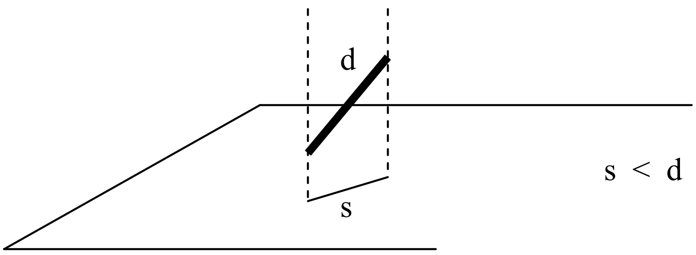
The obtained cut off and extended rectangles will be again equal!

The reason lies in a new feature "compression".

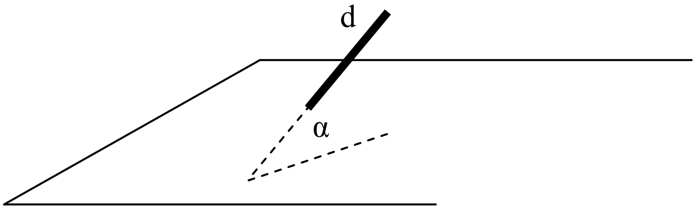
When the sun creates shadows, they sometimes are bigger than the real sizes:



If the sun is above us, say around noon, then all shadows are compressed smaller sizes:



The compression ratio of s from d only depends on d's angle:



All distances with such α angle will have same $\underline{\alpha}$ compression: $s = d \bullet \underline{\alpha}$

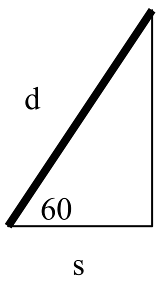
At $\alpha = 0$ the d distance is parallel to the ground, so the s shadow is the same as d.

In other words, $\underline{0} = 1$, there is no compression.

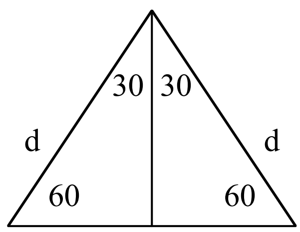
At $\alpha = 90$ the shadow is merely a point that is 0 length, so $\underline{90} = 0$.

From 0 to 90 we will have all possible compressions.

For example, half compression is obtained at $\alpha = 60$.



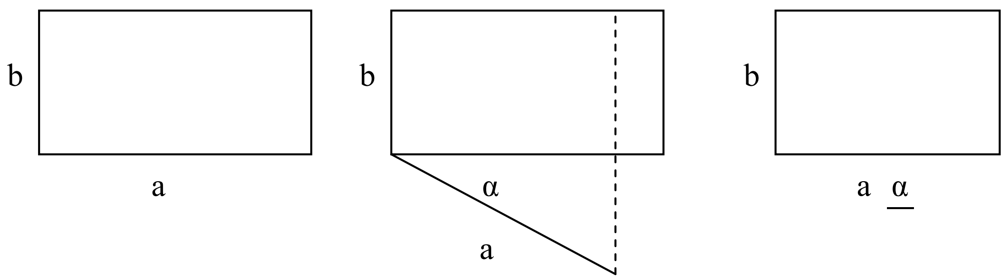
This is so, because the other angle being 30 and mirroring it will make a triangle with all angles being 60:



But then, since all angles are equal, the sides must be equal too, so the base is d too.

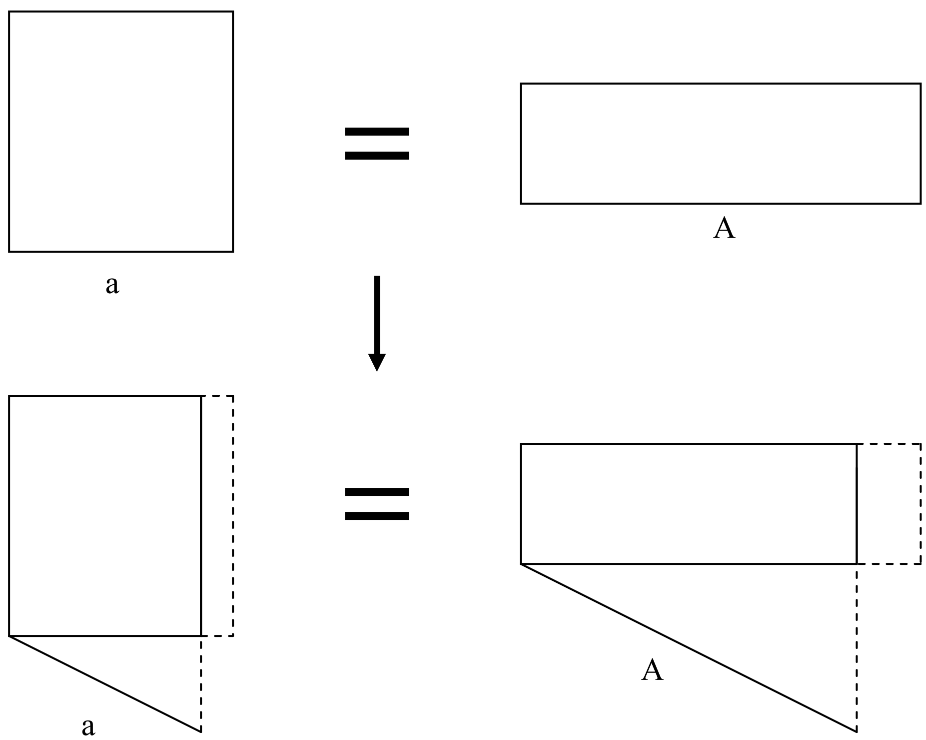
Thus, s is indeed $\frac{d}{2}$.

Now comes an obvious, but crucial fact. If we compress a side of a rectangle, but keep the other side, then its area compresses by the same rate:

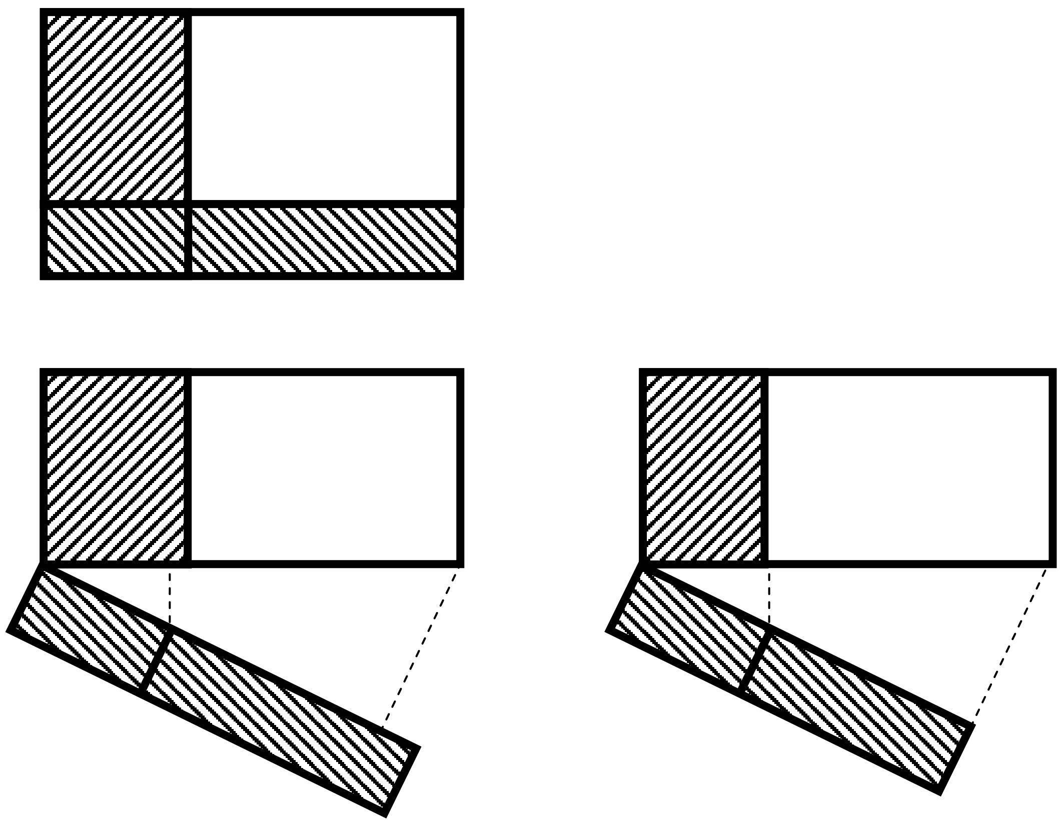


The area of the original is $a\,b$, but of the compressed is $a\,\underline{\alpha}\,b = a\,b\,\underline{\alpha}$.

This also means that if two rectangles are equal and we compress both only in one side, with the same angle, then they remain equal:



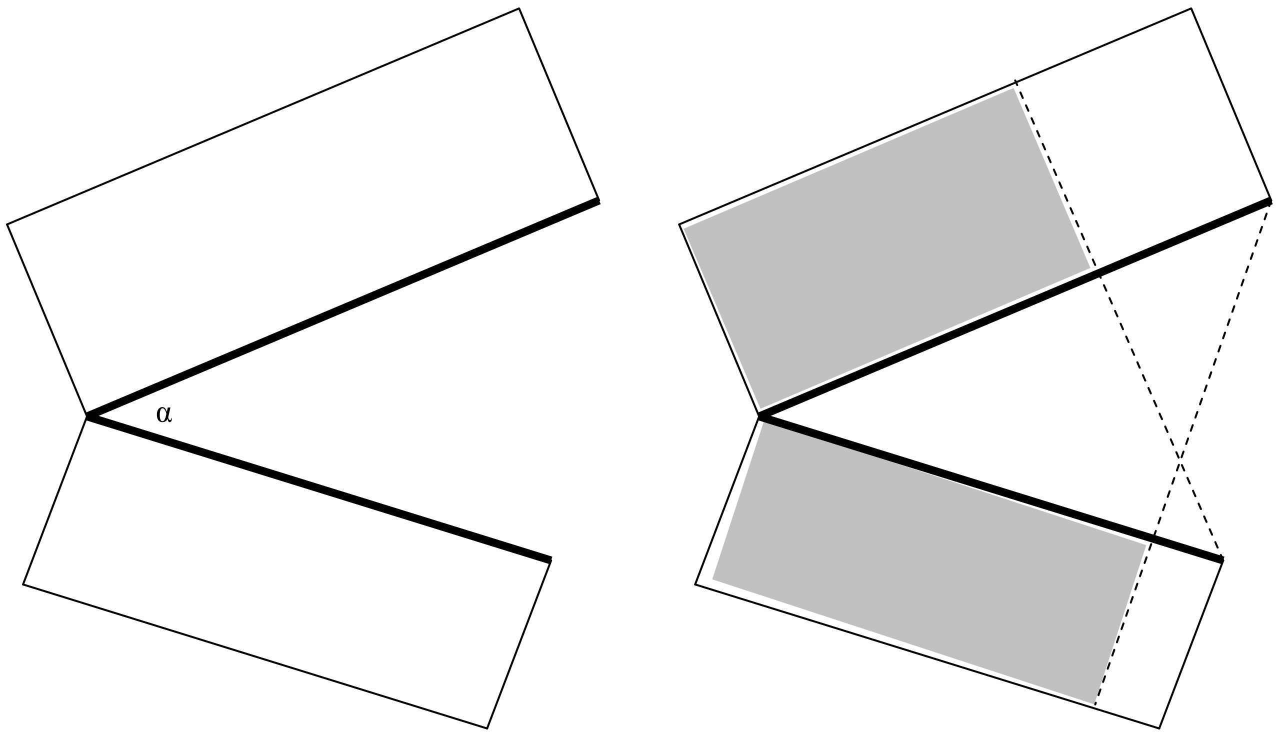
But this is exactly what happened when we turned similar rectangles:



Both compressions were done by $\underline{\alpha}$ corresponding to the α angle between the two similar original rectangles. Of course, the two compressions happened towards the two different sides of the α angle, but this is immaterial, only α counts.

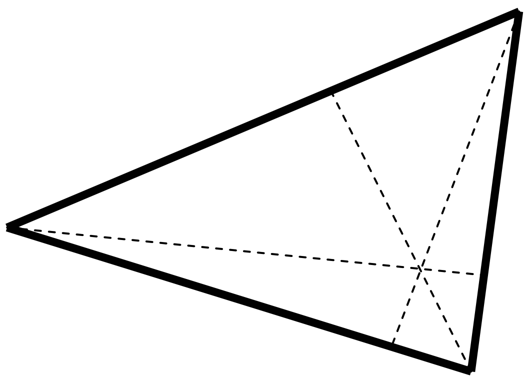
Recapping our result:

Putting similar rectangles on two α angled distances, and then cutting or extending them from the end points of the distances, we get equal rectangles.

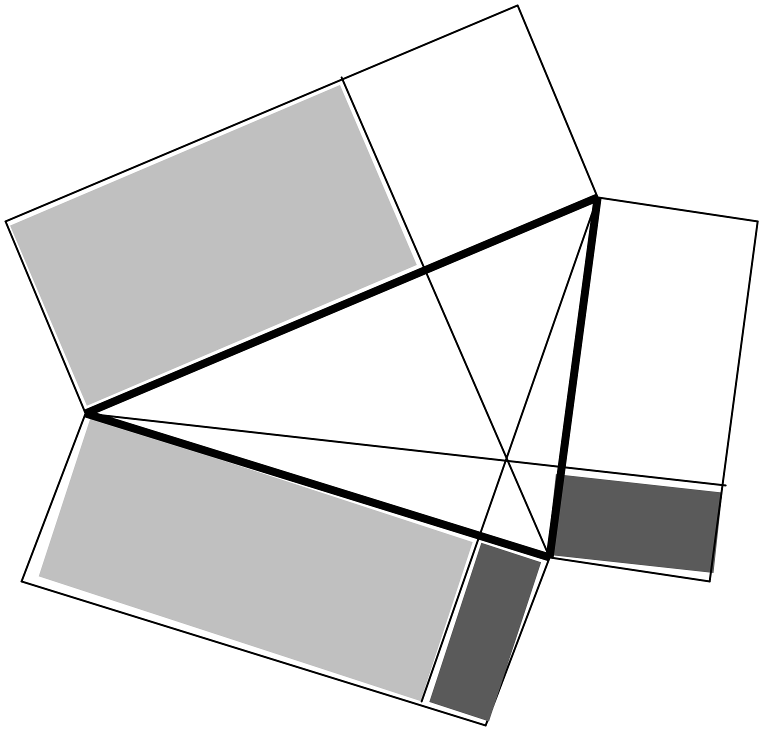


As we see, it can happen that both are cuts so none of them extensions. This merely depends on whether the shadows are in the original distances.

The two distances determine a third across, and if this triangle has all angles less than 90, then all shadows are inside the distances:

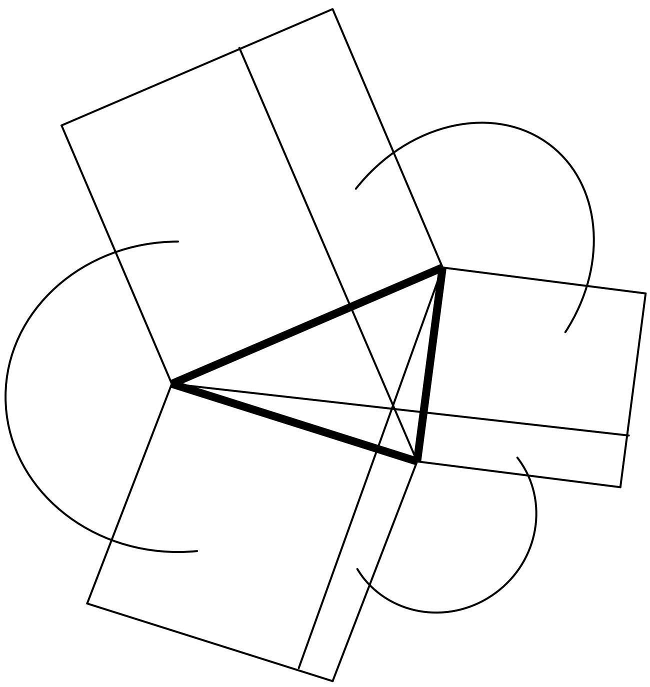


So then putting similar rectangles on each side and cutting them accordingly, we'll cut all three rectangles into two and all corner touching pieces will be equal:

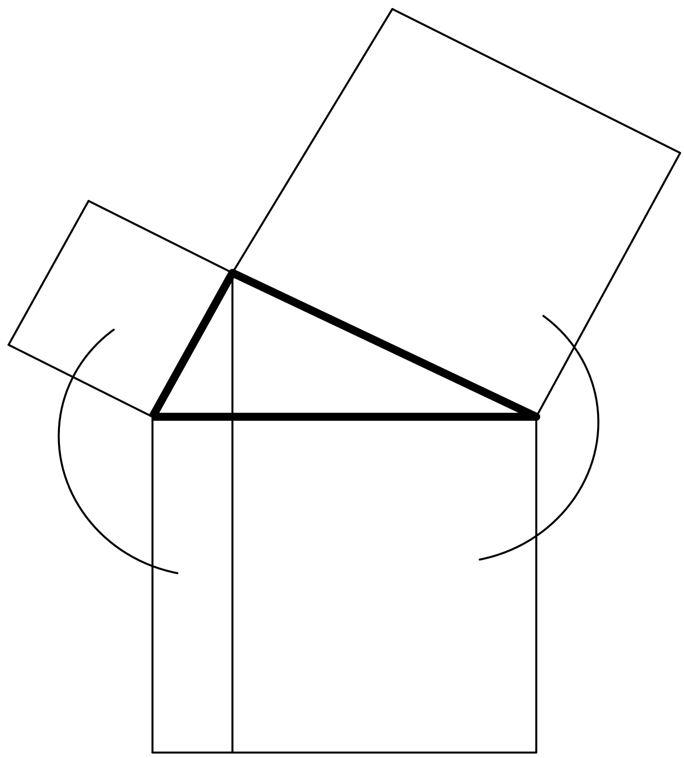


The corner touching equal rectangles are shaded the same.

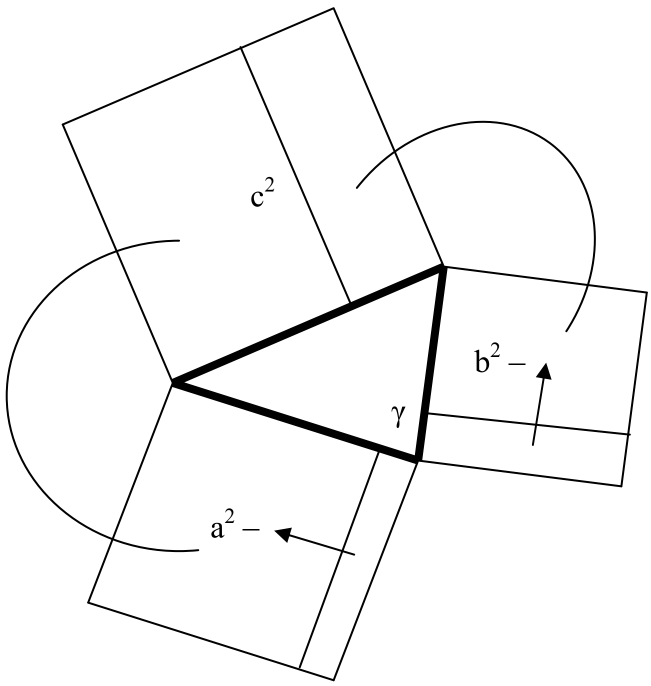
The proportional rectangles must be placed on the sides, so that the sides correspond to each other. In other words, the other sides of the rectangles are a fixed proportion of the sides. The easiest way to ensure this fixed proportion is to make it 1, that is to put squares on the sides.



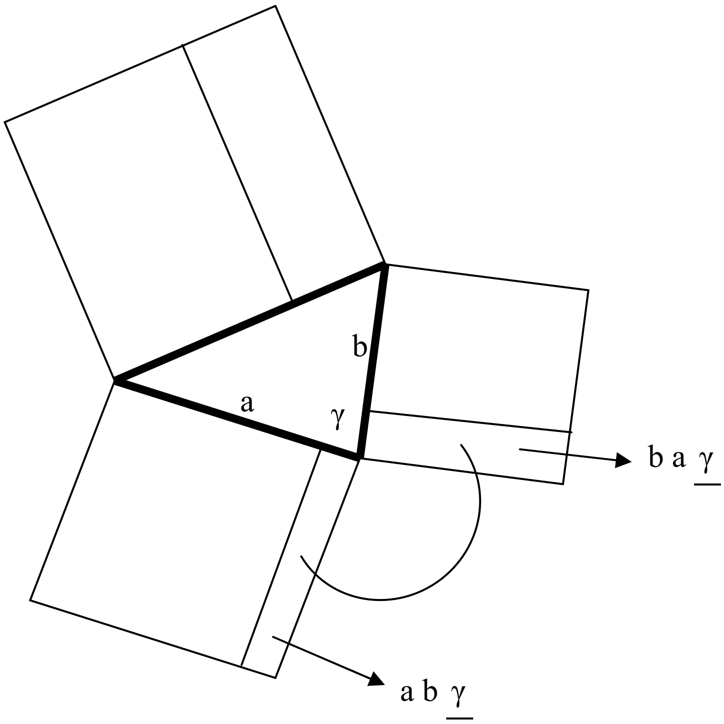
If one of the angles of the triangle is 90, then in two of the squares, we have only a single piece, because the other becomes a line. Thus, the two squares are each equal to the two corner touching pieces in the third square:



This of course is exactly the Pythagoras theorem with the added detail of how the a^2 and b^2 are not only adding up to c^2 but equal to the parts cut by the height line. Amazingly, the general triangular theorem about the square parts can be used in a new direction. One square, say c^2 can be expressed as the total of the two corner touching pieces in a^2 and b^2. So then, $c^2 = a^2 + b^2$ minus the two opposite pieces in a^2 and b^2



Of course, the opposite pieces are equal, namely one is $a\,b\,\underline{\gamma}$, the other $b\,a\,\underline{\gamma}$.



So actually $c^2 = a^2 + b^2 - 2\,a\,b\,\underline{\gamma}$.

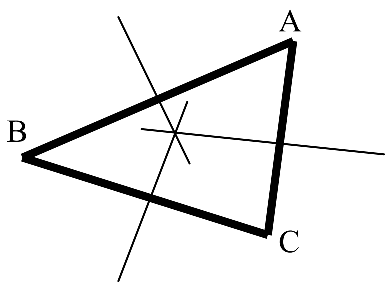
This is the so called Cosine Theorem, if someone calls the compression as cosine.

Finally, there is only one detail we didn’t explain.

Why did we draw the three shadow projection lines or “heights” of the triangle crossing at a single point?

This can be seen as follows:

First we can realize that instead of drawing perpendiculars from across the corners, we can draw one through each center of the sides.

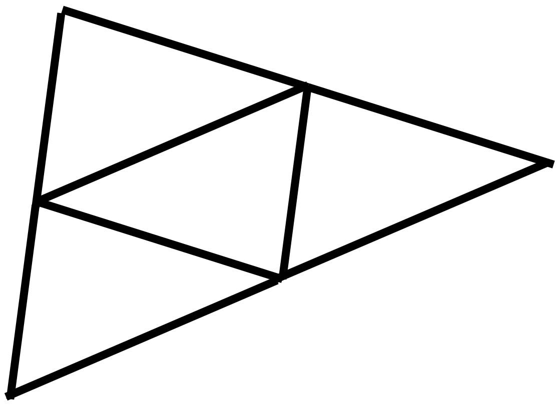


Now these three “middle perpendiculars” have to cross in a single point for the following simple reasons:

Suppose A , B , C , are the three corners.

The middle perpendicular of AB is a line that contains the points equal far from A and B. The middle perpendicular of BC is a line that contains the points equal far from B and C. Then, the crossing point of these two lines must be a point, equal far from A and B and also from B and C. In short, equal distanced from all three points. Thus, this point must be on the middle perpendicular of AC too, since this line contains all points equal far from A and C.

Knowing that the middle perpendiculars cross in one point, we can show the same for the heights. Indeed, we can copy a triangle onto its sides, so that itself and the three copies give a doubled triangle:



And voila, the heights of the original small triangle became the middle perpendiculars of the big one. Thus, they have to cross in one point.