Inequalities Of Means

D The A arithmetical, G geometrical and H harmonic means of some $x_1, x_2, \dots, x_n$ numbers are defined as:

$$A = \frac{x_1 + \dots + x_n}{n} \qquad G = \sqrt[n]{x_1 \dots x_n}$$

$$H = \frac{1}{\dfrac{\frac{1}{x_1} + \dots + \frac{1}{x_n}}{n}} = \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}$$

T If $x_1, x_2, \dots, x_n$ are all ≥ 0 then $A \geq G \geq H$

P First we'll show $A \geq G$, using induction from $n-1$ many numbers to n many.
The initial case is the trivial $A = G$ equality for a single x_1 member.
In the induction step we'll also assume that the members are increasingly named by the variables. Not strictly of course because some can be equal.
The point is that no member is bigger than x_n.
The assumed inequality for the $n-1$ many numbers is:

$$a = \frac{x_1 + \dots + x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1 \dots x_{n-1}} = g$$

Then let $\varepsilon = \dfrac{\frac{x_n}{g} - 1}{n}$ and so: $x_n = g(1 + n\varepsilon)$.
By the assumption that no member is bigger than x_n :

$$x_n = \sqrt[n-1]{x_n^{\,n-1}} \geq \sqrt[n-1]{x_1 \dots x_{n-1}} = g \quad \text{so} \quad \varepsilon \geq 0 \text{ too.}$$

We'll use a lemma for such ε numbers, namely that: $1 + \varepsilon \geq \sqrt[n]{1 + n\varepsilon}$
This is trivial by taking the n-th power, because: $(1 + \varepsilon)^n = 1 + n\varepsilon + \dots$
The proof of $A \geq G$ itself is then:

$$A = \frac{x_1 + \dots + x_n}{n} = \frac{a(n-1) + g(1 + n\varepsilon)}{n} \geq \frac{g(n-1) + g(1 + n\varepsilon)}{n} =$$

$$g(1 + \varepsilon) \geq g\sqrt[n]{1 + n\varepsilon} = g\sqrt[n]{\frac{x_n}{g}} = \sqrt[n]{g^n \frac{x_n}{g}} = \sqrt[n]{g^{n-1} x_n} = G$$

Finally, to show $G \geq H$, we use $A \geq G$ for the $\frac{1}{x_1} \dots \frac{1}{x_n}$ numbers:

$$\frac{\frac{1}{x_1} + \dots + \frac{1}{x_n}}{n} \geq \sqrt[n]{\frac{1}{x_1} \dots \frac{1}{x_n}} = \frac{1}{\sqrt[n]{x_1 \dots x_n}}$$

Taking the reciprocal of the sides proves the claim.

R I discovered the crucial $A \geq G$ part of the previous proof in high school, after I was not happy with the proof presented to me by my teacher. That one used induction to the $2, 4, 8, \ldots$ cases and then implying the others by simply realizing that:

T The $A \geq G$ n member case implies the $a \geq g$ $n-1$ member case.

P We simply regard an $x_n = a$ member and then first of all: $A =$

$$\frac{x_1+\ldots+x_n}{n} = \frac{x_1+\ldots+x_{n-1}+\dfrac{x_1+\ldots+x_{n-1}}{n-1}}{n} = \frac{(x_1+\ldots+x_{n-1})(1+\dfrac{1}{n-1})}{n} =$$

$$\frac{(x_1+\ldots+x_{n-1})\dfrac{n}{n-1}}{n} = \frac{x_1+\ldots+x_{n-1}}{n-1} = a \text{ too!}$$

Then using $A \geq G$ with $x_n = a$ we also have:

$a = A \geq G = \sqrt[n]{x_1 \ldots x_{n-1}\, a}$ which raised to the n-th power gives:

$a^n \geq x_1 \ldots x_{n-1}\, a$ which divided by a and taking $n-1$ root gives:

$$a \geq \sqrt[n-1]{x_1 \ldots x_{n-1}} = g$$

T The mentioned initial case for two members can be seen as:

P

$$(x_1 - x_2)^2 \geq 0 \rightarrow x_1^2 + x_2^2 - 2\, x_1 x_2 \geq 0 \rightarrow$$

$$x_1^2 + x_2^2 + 2\, x_1 x_2 \geq 4\, x_1 x_2 \rightarrow \frac{(x_1+x_2)^2}{4} \geq x_1 x_2 \rightarrow$$

$$\frac{x_1+x_2}{2} \geq \sqrt{x_1 x_2}$$

T From this it's easy to go to the $n = 4$ case:

P

$$\frac{x_1+x_2+x_3+x_4}{4} = \frac{\dfrac{x_1+x_2}{2}+\dfrac{x_3+x_4}{2}}{2} \geq \sqrt{\frac{x_1+x_2}{2}\ \frac{x_3+x_4}{2}} \geq$$

$$\sqrt{\sqrt{x_1 x_2}\ \sqrt{x_3 x_4}} = \sqrt[4]{x_1 x_2 x_3 x_4}$$

Similarly we can jump to $n = 8$, then $n = 16$ and so on.