Infinite Sums

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1. Powers

T

1.) $\frac{1}{2} + \frac{1}{4} + \ldots\ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

2.) $\frac{1}{2} + \frac{1}{4} + \ldots\ldots + \frac{1}{2^n} + \ldots\ldots\ldots = 1$

3.) $\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots\ldots\ldots = \frac{1}{2^n}$

4.) $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots\ldots + \frac{n}{2^n} + \ldots\ldots\ldots = 2$

5.) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots\ldots = \frac{1}{3}$

6.) $\frac{1}{2}\,,\ \frac{1}{4}\,,\ \frac{3}{8}\,,\ \frac{5}{16}\,,\ \frac{11}{32}\,,\ \ldots\ldots\ a_{n+1} = \frac{a_n + a_{n-1}}{2} \rightarrow \frac{1}{3}$

P

1.) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots + \frac{1}{2^{n-1}} + \frac{1}{2^n} =$

$$\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{8}\right) + \ldots\ldots + \left(\frac{1}{2^{n-2}}-\frac{1}{2^{n-1}}\right) + \left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right) = 1 - \frac{1}{2^n}$$

0 0 0 0 0

2.) $\frac{1}{2^n} \rightarrow 0$ Or using the trick in 1.) for infinite many terms:

$$\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{8}\right) + \ldots\ldots\ldots = 1$$

3.) $\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots\ldots = 1 - \left(\frac{1}{2}+\frac{1}{4}+\ldots.+\frac{1}{2^n}\right) = 1 - \left(1-\frac{1}{2^n}\right) = \frac{1}{2^n}$

Or using the same trick as in 1.) and 2.) , term by term:

$$\left(\frac{1}{2^n}-\frac{1}{2^{n+1}}\right) + \left(\frac{1}{2^{n+1}}-\frac{1}{2^{n+2}}\right) + \ldots\ldots\ldots = \frac{1}{2^n}$$

4.) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots\ldots = 1$

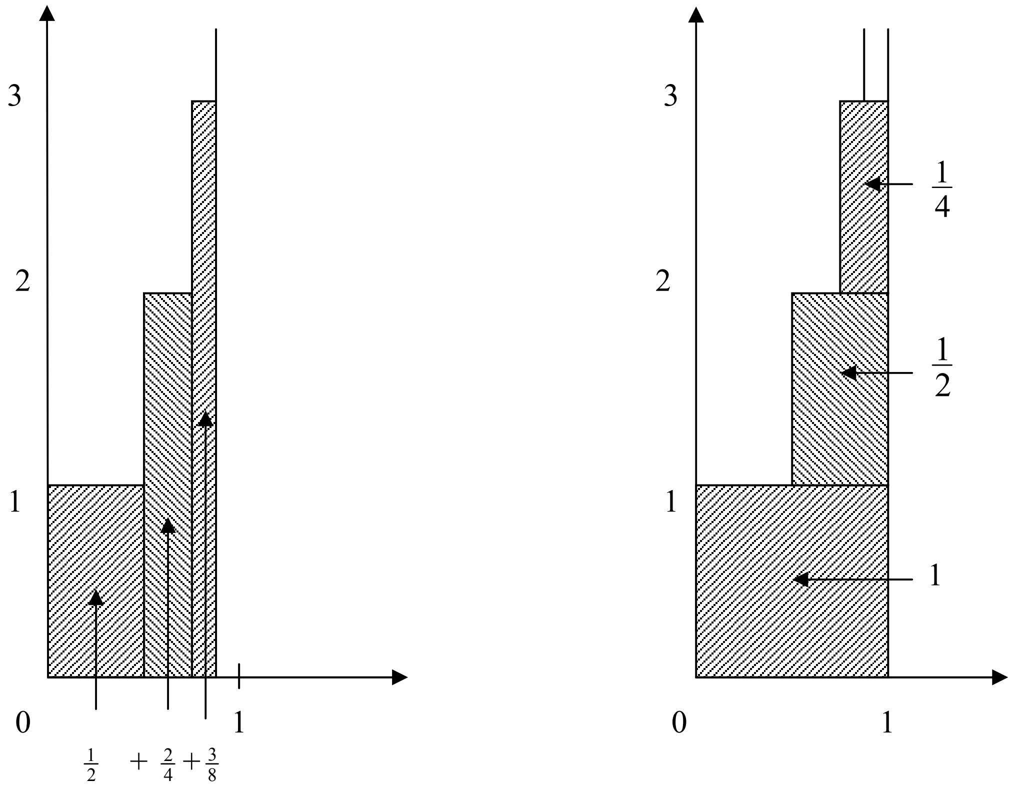
$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots\ldots = \frac{1}{2}$

$\frac{1}{8} + \frac{1}{16} + \ldots\ldots\ldots = \frac{1}{4}$

Added up, column by column:

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots\ldots = 1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots = 1 + 1 = 2$$

We can give a nice visualization of this in the x, y coordinate system with x as denominator and y as numerator:



5.) $\left[\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots\ldots\right] = \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots\ldots\right) - \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\ldots\right) =$

$2\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\ldots\right) - \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\ldots\right) = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\ldots$

Thus $2\,[\quad] = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots\ldots$ and so

$3\,[\quad] = 2\,[\quad] + [\quad] = \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots\ldots\right) + \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\ldots\right) =$

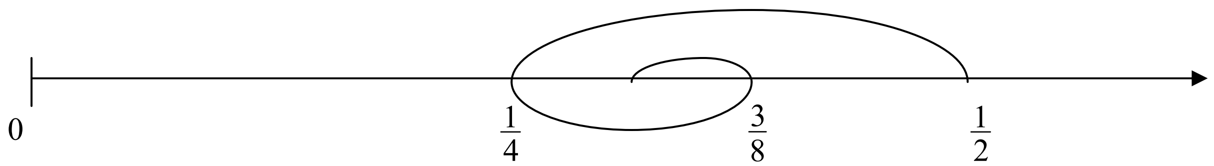
$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots = 1$ So $[\quad] = \frac{1}{3}$

6.) $a_{n+1} = \frac{a_n + a_{n-1}}{2} \quad / - a_n$

$a_{n+1} - a_n = -\frac{a_n - a_{n-1}}{2}$ .Using d_n for the differences: $d_{n+1} = -\frac{d_n}{2}$ with $d_1 = a_1$ and

$a_n = d_1 + d_2 + \ldots + d_n$ so $a_n \rightarrow d_1 + d_2 + \ldots = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots\ldots = \frac{1}{3}$

This means that our original a_n sequence is alternating to the non obvious $\frac{1}{3}$ limit:



T

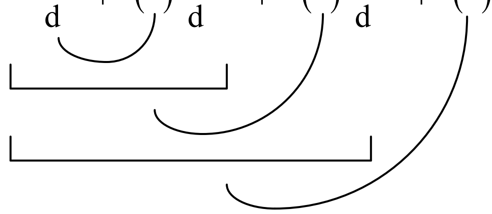
1.) If p is positive then $(1+p)^n \to \infty$
If $0 < q < 1$ then $q^n \to 0$

2.) If $0 < q < 1$ then $1 + q + q^2 + \dots + q^n = \frac{1-q^{n+1}}{1-q}$

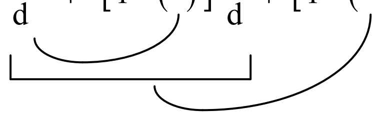
3.) If $0 < q < 1$ then $1 + q + q^2 + \dots + q^n + \dots = \frac{1}{1-q}$

4.) If $d > 1$ then $\frac{1}{d} + \frac{1}{d^2} + \dots + \frac{1}{d^n} + \dots = \frac{1}{d-1}$

5.) $\frac{1}{d} + (\;)\frac{1}{d} + (\;)\frac{1}{d} + (\;)\frac{1}{d} + \dots\dots = \infty$



6.) $\frac{1}{d} + [1-(\;)]\frac{1}{d} + [1-(\;)]\frac{1}{d} + \dots\dots = 1$

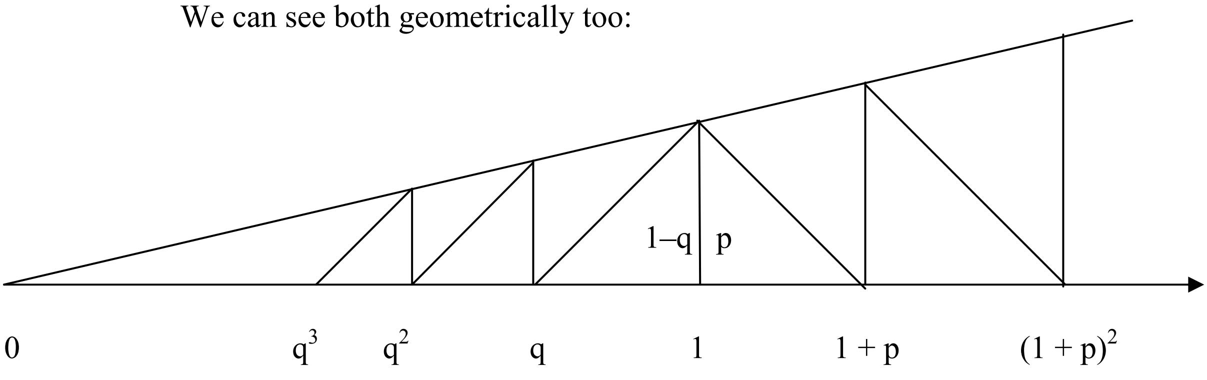


P

1.) $(1+p)^n = 1 + np + \dots > np \to \infty$

$$q^n = \frac{1}{\left(1 + \frac{1-q}{q}\right)^n} = \frac{1}{(1+p)^n} \to \frac{1}{\infty} = 0$$

We can see both geometrically too:



2.) Multiplying the left with $(1-q)$ term by term:

$$(1-q) + (1-q)\,q + (1-q)\,q^2 + \dots + (1-q)\,q^n =$$

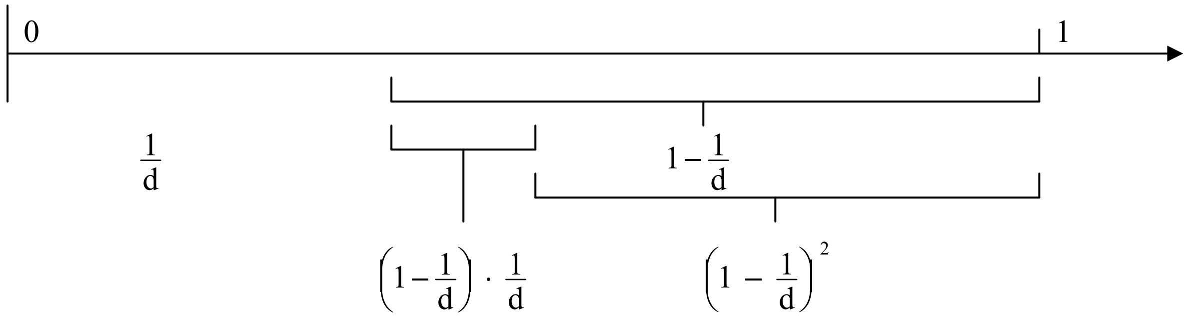
$$1 - \cancel{q} + \cancel{q} - \cancel{q^2} + \cancel{q^2} - \cancel{q^3} + \dots + \cancel{q^n} - q^{n+1} = 1 - q^{n+1}$$

3.) Follows from 1.) and 2.) : $\frac{1-q^{n+1}}{1-q} \to \frac{1-0}{1-q} = \frac{1}{1-q}$

4.) $\frac{1}{d} + \frac{1}{d^2} + \ldots. = 1 + \frac{1}{d} + \frac{1}{d^2} + \ldots. - 1 = \frac{1}{1-\frac{1}{d}} - 1 = \frac{d}{d-1} - 1 = \frac{1}{d-1}$

5.) $\frac{1}{d} + (\ )\frac{1}{d} + (\ )\frac{1}{d} + \ldots + (\ )\frac{1}{d} = \frac{1}{d}\left(1+\frac{1}{d}\right)\left(1+\frac{1}{d}\right) \ldots. \left(1+\frac{1}{d}\right) = (1+p)^n \to \infty$

6.) While in 3.) the reached sum was divided by d and added, here the remaining distance to 1 is divided by d and added. Thus the remaining distance always becomes $\left(1-\frac{1}{d}\right)$ part of the previous:



This of course approaches 0 by 1.)

R The fact that the limit in 1.) and the sum in 6.) is independent of q and d is not a paradox but it has some strange consequences.
For example if we ask which decimal numbers are more between 0 and 1, those that have a chosen digit say “7” or those that don’t, then it might seem natural that those that don’t. After all we only exclude one digit so we can still use all the nine others. But strangely excluding the digit “7” actually eliminates “almost all” numbers. This becomes apparent if we collect those numbers that have “7” as first decimals, in other words look like 0.7… These are only one tenth of all numbers between 0 and 1 and so nine tenth of them remain. Then the ones with “7” on the second decimal are again 0.1-th of these and so 0.9-th remain.
Thus in the end 0.9 x 0.9 x .... part remains which tends to 0 by 1.)

2. Squares and factorials Irrational sum

D

$n\,! \;:=\; 2\cdot 3\cdot 4\cdot \ldots\ldots \cdot n$

T

1.) $\frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots\ldots + \frac{1}{n\,!} + \ldots\ldots < 1$

2.) $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots + \frac{1}{n^2} + \ldots\ldots < 1$

3.) $\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \ldots\ldots\ldots < \frac{1}{n}$

4.)
$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots < \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots\ldots\ldots$

5.) $\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots\ldots\ldots < \frac{1}{n\cdot n\,!}$

6.) $\frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots\ldots + \frac{1}{n\,!} + \ldots\ldots =$ irrational

P

1.)
$$\frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots\ldots$$
$$\wedge \qquad \wedge$$
$$\frac{1}{2\cdot 2} \qquad \frac{1}{2\cdot 2\cdot 2}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots = 1$$

2.)
$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \ldots\ldots\ldots$$
$$\frac{1}{3^2} < \frac{1}{2^2}, \quad \frac{1}{5^2} < \frac{1}{4^2}, \quad \frac{1}{6^2} < \frac{1}{4^2}, \quad \frac{1}{7^2} < \frac{1}{4^2}, \quad \frac{1}{9^2} < \frac{1}{8^2}, \quad \frac{1}{10^2} < \frac{1}{8^2}$$
$$2\cdot\frac{1}{2^2} \qquad 4\cdot\frac{1}{4^2} \qquad 8\cdot\frac{1}{8^2}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots = 1$$

We give an other proof:
$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$
$$\wedge \qquad \wedge \qquad \wedge$$
$$\frac{1}{1\cdot 2} \qquad \frac{1}{2\cdot 3} \qquad \frac{1}{3\cdot 4}$$
$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots = 1$$

3.)
$$\frac{1}{(n+1)^2} \quad + \quad \frac{1}{(n+2)^2} \quad + \quad \cdots$$
$$\wedge \qquad\qquad \wedge$$
$$\frac{1}{n(n+1)} \qquad \frac{1}{(n+1)(n+2)}$$
$$\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \cdots = \frac{1}{n}$$

4.)
$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < \frac{1}{4} + \frac{1}{9} + \frac{1}{3} = \frac{9+4+12}{36} = \frac{25}{36} = \frac{50}{72}$$
$$\wedge$$
$$\frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots > \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{12+4+1}{24} = \frac{17}{24} = \frac{51}{72}$$

5.)
$$\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots\right) <$$
$$\frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right) = \frac{1}{n!}\cdot\frac{1}{(n+1) - 1} = \frac{1}{n!}\cdot\frac{1}{n}$$

6.) Suppose $\frac{1}{2} + \frac{1}{2\cdot 3} + \cdots = \frac{a}{b}$ and let $n > b$. Then calculate:

$$n!\left[\frac{a}{b} - \left(\frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n!}\right)\right] = n!\left[\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right] < \frac{1}{n}$$
$$\vee \qquad\qquad\qquad \wedge$$
$$0 \qquad\qquad\qquad \frac{1}{n!}\cdot\frac{1}{n}$$

$n!\left[\frac{a}{b} - \left(\frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n!}\right)\right]$ is whole because $b < n$ and so every denominator is a factor of $n!$.

Thus we got a clear contradiction that a whole is smaller than $\frac{1}{n}$.

R

3.) and 5.) compared shows that the end section of the factorial reciprocals is much smaller than that of the square reciprocals. Yet 4.) shows that the total sum is opposite. This happens simply because at the beginning, namely for the first two members only, the reciprocal of the factorials were bigger. So the value of an infinite sum doesn't necessarily reflect how fast the numbers are increasing because some beginning differences cannot be eliminated. On the other hand the irrationality of the factorial reciprocal sum is a consequence of the extremely slow increase of it.

3. Slow infiniteness Subset paradoxes

T

1.) The smallest common multiple of $2, 3, \ldots\ n$ is $C = 2^{m_1} \cdot 3^{m_2} \cdot \ldots \cdot p_k^{m_k}$ with:

a.) $2, 3, \ldots\ p_k$ are all the primes up to n

b.) $p_i^{m_i}$ is the biggest power of p_i up to n.

2.) If we bring $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ to common denominator with the C of 1.) as: $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \frac{A}{C}$ then:

a.) A is odd

b.) $\frac{A}{C}$ is never whole

P

1.) Trivial as the following example shows:
For $2, 3, \ldots\ 10$ $C = 2^3 \cdot 3^2 \cdot 5 \cdot 7$ and all these primes and powers are needed.

2.) a.) In the addition with C denominators there will be only one numerator odd namely the one that had 2^{m_1} denominator. For example: $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{10} =$

$$\frac{2^2 \cdot 3^2 \cdot 5 \cdot 7 + 2^3 \cdot 3 \cdot 5 \cdot 7 + 2 \cdot 3^2 \cdot 5 \cdot 7 + 2^3 \cdot 3^2 \cdot 7 + 2^2 \cdot 3 \cdot 5 \cdot 7 + 2^3 \cdot 3^2 \cdot 5 + \overline{3^2 \cdot 5 \cdot 7} + 2^3 \cdot 5 \cdot 7 + 2^2 \cdot 3^2 \cdot 7 + \ldots}{2^3 \cdot 3^2 \cdot 5 \cdot 7}$$

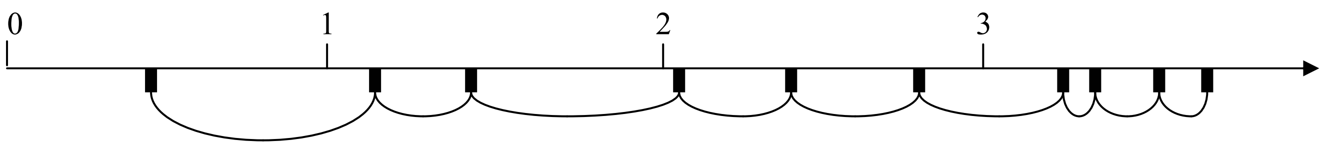
The marked term in the numerator is the only one without factor 2, so the numerator is odd.

b.) Trivial by a.)

R

1.)
In the above example the 3 and 7 prime factors are also missing in only one numerator, so A is not dividable by these either. But 5 is missing from two so we can't say that A has only bigger prime factors than 10. With more detailed examination though the aboves can be continued to show that $\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ is not only never whole but is a more and more complicated fraction as n grows. The size of these sums is of course more important than their exact values so we turn to this question now.
2.)
All of our sums up until now were less than 1 and were all subsets of the full $\frac{1}{2}+\frac{1}{3}+\ldots$
This clearly shows that this full sum of reciprocals must be much bigger than 1 but it still might surprise us that the full sum is actually infinite. This paradox is an example of the divergent but "self approaching" sequences that I would call the anti-Achilles paradox. It was only Cauchy who first cleared it up completely. Here I'll sketch the main ideas:

3.)
The fact that the sum of infinite many numbers can be finite is obvious by looking at $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots = 1$ on the $(0,1)$ distance line. But the same fact regarded in time seemed paradoxical to the greeks. Indeed if Achilles is twice as fast as an other runner and so he gives a mile headstart, he should still catch up with his challenger. Yet we can "refute" this with the following argument: When Achilles finishes the 1 mile headstart he gave then his opponent will be half a mile further ahead. When Achilles finishes this then he will be a quarter of a mile ahead again. Then an eight of a mile ahead, and so on always ahead. So we might easily jump to the conclusion that Achilles never catches up. To see the fault in this conclusion is enough to realize that every event in time had infinite many instances previously. For examples if the rain starts to fall then an hour ago it wasn't raining yet. Half an hour ago it still wasn't raining. Similarly a quarter, an eight, and so on fractions of the hour back. So it wasn't raining infinite many times and yet it didn't stay without rain forever.
After we get used to the idea of smaller and smaller distances leading to a finite sum, we might jump to the other extreme and think that adding up smaller and smaller distances must always be finite. But this is not true and we could call its refutation as the anti-Achilles paradox:
Indeed all we have to do is put more and more points between the increasing whole numbers on the infinite number line.



The differences between these denser and denser points can clearly be smaller and smaller. And yet the total sum of the differences up to a point is equal to the distance between that point and the first point after zero and so obviously it gets arbitrary large.
Using the whole numbers themselves and equally distributed points between them, one on (0,1), two on (1,2), and so on, we can even exactly calculate the whole numbers as the sum of decreasing differences, and so we get an algebraically also obvious example:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} + \ldots + \frac{1}{n} = n - 1$$

By all these it should be obvious that smaller and smaller numbers can add up to infinity. Then it's only the simplicity of $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ that is surprising.

T

1.) $\frac{1}{n+1} + \frac{1}{n+2} + \ . \ . \ . \ . \ + \frac{1}{2n} > \frac{1}{2}$

2.) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ . \ . \ . \ . \ . \ . \qquad = \infty$

3.) $\frac{1}{N} + \frac{1}{2N} + \frac{1}{3N} + \ . \ . \ . \ . \ . \ . \qquad = \infty$

4.) $\frac{1}{N+D} + \frac{1}{2N+D} + \frac{1}{3N+D} + \ . \ . \ . \ . \ . \ . \qquad = \infty$

5.) $\sum \frac{1}{\text{evens}} = \frac{1}{2} + \frac{1}{4} + \ . \ . \ . \ = \infty$, $\sum \frac{1}{\text{odds}} = \frac{1}{3} + \frac{1}{5} + \ . \ . \ . \ = \infty$

6.) $\sum \frac{1}{\text{composites}} = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \ . \ . \ . \ . \ = \infty$

P

1.) $\frac{1}{n+1} + \frac{1}{n+2} + \ . \ . \ . \ . \ + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \ . \ . \ . \ . \ + \frac{1}{2n} = n \cdot \frac{1}{2n} = \frac{1}{2}$

2.) By 1.) from any n-th member of the sum we can go up to the 2 n-th and thus increase the value by more than $\frac{1}{2}$. Repeating this we can guarantee infinite many, more than $\frac{1}{2}$ segments, thus giving arbitrary big sum.
We can give a self contained proof with the same argument but starting from the beginning and repeating the proof of 1.) :

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \ . \ . \ . \ . \ . \ .$$

$$\frac{1}{3} > \frac{1}{4}, \qquad \frac{1}{5} > \frac{1}{8},\ \frac{1}{6} > \frac{1}{8},\ \frac{1}{7} > \frac{1}{8}, \qquad \frac{1}{9} > \frac{1}{16},\ \frac{1}{10} > \frac{1}{16}$$

$$\underbrace{\frac{1}{3} + \frac{1}{4}}_{\frac{1}{2}} \qquad \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\frac{1}{2}} \qquad \underbrace{\frac{1}{9} + \frac{1}{10} + \dots}_{\frac{1}{2}}$$

3.) $\frac{1}{N} + \frac{1}{2N} + \ . \ . \ . \ . \ = \frac{1}{N}\left(1 + \frac{1}{2} + \frac{1}{3} + . \ . \ . \ .\right) = \frac{1}{N} \cdot \infty$

4.) $\frac{1}{N+D} + \frac{1}{2N+D} + \ . \ . \ . \ = \frac{1}{N}\left(\frac{1}{1+\frac{D}{N}} + \frac{1}{2+\frac{D}{N}} + \ . \ . \ .\right) > \frac{1}{N}\left(\frac{1}{n+1} + \frac{1}{n+2} + . \ . \ .\right) = \infty$

where n is the next whole number after $\frac{D}{N}$ or itself $\frac{D}{N}$ if it is a whole.

5.) For evens follows from 3.) with N = 2, for odds from 4.) with N = 2 and D = 1.

6.) Trivial by 3.) .

R 6.) might suggest that the sum of prime-reciprocals $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\dots$ is finite by the following "reasoning": The composites are the union of infinite many sets each containing the multiples of different primes. Each of these gives an infinite sum of reciprocals and though these sets are not disjoint they don't have "too many" common members. So the sum of composite-reciprocals is "infinite times infinite". Thus taking off this much from the total reciprocals should leave only a finite sum. Of course the error is that from an infinite we can take off infinite many infinites and still end up with infinite. And indeed the sum of prime-reciprocals is ∞, as we'll show it soon.

A much more paradoxical situation appears if we close out the reciprocals of numbers, containing a chosen digit, say 0. Of course we already showed that such closing out leaves out more decimals then it allows, but it's still surprising that:

T The sum of reciprocals of those numbers that don't contain 0 digit is $<$ 30

P

$$\underbrace{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{9}}_{\wedge\; 3}+\underbrace{\frac{1}{11}+\dots+\frac{1}{19}+\frac{1}{21}+\dots+\frac{1}{99}}_{\wedge\;\frac{1}{10}\,\dots\,\frac{1}{10}\;\frac{1}{20}\,\dots\,\frac{1}{90}\;\to\;\wedge\;\frac{3\cdot 9}{10}}+\underbrace{\frac{1}{111}+\dots+\frac{1}{999}}_{\wedge\;\frac{1}{100}\,\dots\,\frac{1}{900}\;\to\;\wedge\;\frac{3\cdot 9\cdot 9}{100}}+\underbrace{\frac{1}{1111}+\dots}_{\wedge\;\frac{1}{1000}\,\dots\;\to\;\wedge\;\frac{3\cdot 9^3}{10^3}}$$

$$=\;3\left[1+\frac{9}{10}+\left(\frac{9}{10}\right)^2+\dots\right]=3\left[\frac{1}{1-\frac{9}{10}}\right]=30$$

As can be seen, we rounded all denominators to the first digit and every rounded block contained 9 times more new members.

4. Associated products

T If $x_1, x_2, \ldots$ are non negative and less than 1 then:

1.) $(1 + x_1) \ldots (1 + x_n) > x_1 + \ldots + x_n$

$(1 - x_1) \ldots (1 - x_n) \le \dfrac{1}{(1+x_1)\ldots(1+x_n)}$

$x_1 + \ldots + x_n \ge 1 - (1 - x_1) \ldots (1 - x_n)$

2.) $\sum x = x_1 + x_2 + \ldots = \infty$

$\prod (1 + x) = (1 + x_1)(1 + x_2)\ldots = \infty$

$\prod (1 - x) = (1 - x_1)(1 - x_2)\ldots = 0$

imply each other and since the first two are monotone increasing and the last decreasing:

$\sum x = S, \ \prod (1 + x) = P, \ \prod (1 - x) = P > 0$ also imply each other.

P 1.) $>$ is trivial because multiplying the left side gives the right and other non negative terms.

$\le$ is also trivial term by term because:

$(1 - x)(1 + x) = 1 - x^2 \le 1$ so $1 - x \le \dfrac{1}{1+x}$

$\ge$ is trivial for $n = 1$ and the $(n-1)$ case multiplied with the positive $(1-x_n)$ and adding x_n gives:

$(x_1 + \ldots + x_{n-1})(1-x_n) + x_n \ge (1-x_n) - (1-x_1)\ldots(1-x_{n-1})(1-x_n) + x_n$

$x_1 + \ldots + x_{n-1} - x_n(x_1 + \ldots + x_{n-1}) \ge 1 - (1-x_1)\ldots(1-x_n)$

Leaving out the $-x_n(x_1 + \ldots + x_{n-1})$ term we get the n case.

2.) Enough to show that:

$\sum x = \infty \Rightarrow \prod(1+x) = \infty \Rightarrow \prod(1-x) = 0 \Rightarrow \sum x = \infty$

The first is direct consequence of $>$ in 1.), the second of $\le$ in 1.).

The third is a consequence of $\ge$ in 1.) but not directly:

If $\prod(1-x) = 0$ then for some n_1 : $(1-x_1)\ldots(1-x_{n_1}) < \frac{1}{2}$ and so:

$1 - (1-x_1)\ldots(1-x_{n_1}) > \frac{1}{2}$ and so by $\ge$ in 1.) $x_1 + \ldots + x_{n_1} > \frac{1}{2}$ too.

Then $\prod\limits_{n_1}(1-x) = (1-x_{n_1+1})(1-x_{n_1+2})\ldots = 0$ so for some $n_2 > n_1$:

$(1-x_{n_1+1})\ldots(1-x_{n_2}) > \frac{1}{2}$

And so on we get infinite many $\frac{1}{2}$ valued segments so $\sum x = \infty$

R

1.)
We give two basic examples, namely the associated products for the

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots = \infty$ and $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots < 1$ sums:

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\ldots\left(1+\frac{1}{n}\right)\ldots\ldots\ldots\ldots = \infty$$
$$\frac{3}{2} \qquad \frac{4}{3} \qquad \frac{5}{4} \qquad\qquad \frac{n+1}{n} = \frac{n+1}{2} \to \infty$$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\ldots\left(1-\frac{1}{n}\right)\ldots\ldots\ldots\ldots = 0$$
$$\frac{1}{2} \qquad \frac{2}{3} \qquad \frac{3}{4} \qquad\qquad \frac{n-1}{n} = \frac{1}{n} \to 0$$

$$\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{4^2}\right)\ldots\left(1+\frac{1}{n^2}\right)\ldots\ldots\ldots\ldots = P$$

We don’t have any tricks to get this P but for the (1 – x) case:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)\ldots\ldots\ldots\ldots = \frac{1}{2}$$

$$\frac{2^2-1}{2^2} \qquad \frac{3^2-1}{3^2} \qquad \frac{4^2-1}{4^2} \qquad\qquad \frac{n^2-1}{n^2}$$

$$\frac{(2-1)(2+1)}{2\cdot 2}\;\frac{(3-1)(3+1)}{3\cdot 3}\;\frac{(4-1)(4+1)}{4\cdot 4}\;\ldots\;\frac{(n-1)(n+1)}{n\cdot n} = \frac{1\cdot(n+1)}{2n} \to \frac{1}{2}$$

The P value of the $\prod\left(1+\frac{1}{n^2}\right)$ case can be calculated with higher tools and it is $\frac{e^{\pi} - \frac{1}{e^{\pi}}}{4\pi}$

2.)
As we see from the examples, our beautiful theorem is still not perfect because it doesn’t relate the actual limits. Indeed it would be fantastic to be able to calculate the associated products from the sums or vice versa. For example the exact value of $\frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$ was calculated by Euler with years of work and the value is $\frac{\pi^2}{6} - 1$. The associated negative product as we just saw is the simple $\frac{1}{2}$ value. This shows that if there is a correlation then it must be fairly complicated and very different for the (1 + x) and (1 – x) cases.

3.)
Completely different type of relation with "products" arise if we multiply two sums term by term. Multiplying two sums correctly we have to multiply every term with every term. Now if we use "×" as a new "product" by just multiplying every term with only the same indexed one, then we only get a very small part of the real product of the two sums.
$1 + \frac{1}{2} + \frac{1}{3} + \; . \; . \; . \; .$ and $\frac{1}{N+1} + \frac{1}{N+2} + \; . \; . \; . \; .$ are both infinite so just how small is the "×" "product" is shown by the following theorem:

T

$$\sum_0 \times \sum_N = \left(1 + \frac{1}{2} + \frac{1}{3} + . . . \right) \times \left(\frac{1}{N+1} + \frac{1}{N+2} + ... \right) :=$$

$$\frac{1}{1\cdot(N+1)} + \frac{1}{2\cdot(N+2)} + \frac{1}{3\cdot(N+3)} + ... = \frac{1+\frac{1}{2}+\frac{1}{3}+. . .+\frac{1}{N}}{N} = \frac{\sum_0 - \sum_N}{N}$$

P

$$\frac{1}{1\cdot(N+1)} + \frac{1}{2\cdot(N+2)} + \frac{1}{3\cdot(N+3)} + ... + \frac{1}{n\cdot(N+n)} + ... =$$

$$\frac{1}{N}\left(\frac{1}{1} - \frac{1}{N+1}\right) + \frac{1}{N}\left(\frac{1}{2} - \frac{1}{N+2}\right) + \frac{1}{N}\left(\frac{1}{3} - \frac{1}{N+3}\right) + . . . + \frac{1}{N}\left(\frac{1}{n} - \frac{1}{N+n}\right) + . . . =$$

$$\frac{1}{N}\left[1 - \cancel{\frac{1}{N+1}} + \frac{1}{2} - \cancel{\frac{1}{N+2}} + \frac{1}{3} - \cancel{\frac{1}{N+3}} + . . . + \frac{1}{n} - \cancel{\frac{1}{N+n}} + . . . + \frac{1}{N} - \cancel{\frac{1}{2N}} + \cancel{\frac{1}{N+1}} - \cancel{\frac{1}{2N+1}} + . . .\right]$$

$$= \frac{1}{N}\left[1 + \frac{1}{2} + . . . + \frac{1}{N}\right]$$

5. Prime reciprocals Euler's formula

T

Euler's formula

If p_1 , p_2 , ... , p_n are different primes and d_1 , d_2 , . . . are all the different numbers obtainable from p-s as factors, then:

$$1 + \frac{1}{d_1} + \frac{1}{d_2} + \ldots\ldots \leq \frac{1}{1-\frac{1}{p_1}} \cdot \frac{1}{1-\frac{1}{p_2}} \cdot \ldots\ldots \cdot \frac{1}{1-\frac{1}{p_n}}$$

If we accept the unique prime factorization of numbers then equality stands.

P

$$\frac{1}{1-\frac{1}{p}} = 1 + \frac{1}{p-1} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots\ldots \quad \text{because:}$$

$$\left(\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots\right) \cdot \frac{p-1}{p-1} = \frac{1-\frac{1}{p}+\frac{1}{p}-\frac{1}{p^2}+\frac{1}{p^2}-\frac{1}{p^3}+\ldots}{p-1} = \frac{1}{p-1}$$

So all we have to show is that:

$$1 + \frac{1}{d_1} + \frac{1}{d_2} + \ldots \leq \left(1+\frac{1}{p_1}+\frac{1}{p_1^{\,2}}+\ldots\right)\left(1+\frac{1}{p_2}+\frac{1}{p_2^{\,2}}+\ldots\right) \ldots \left(1+\frac{1}{p_n}+\frac{1}{p_n^{\,2}}+\ldots\right)$$

And indeed, multiplying all possible terms on the right, we get all reciprocals of numbers with p_1 , p_2 , . . . p_n prime factors. If factorizations are unique we get exactly those.

T

1.) There are infinite many primes.

2.) $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) \ldots \left(1-\frac{1}{p}\right) \to 0$ where 2, 3, . . . p are all the primes up to a p.

3.) $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p} + \ldots = \infty$

P

1.) If p_1 , p_2 , . . . p_n were all the primes, then the d_1 , d_2 , . . . numbers obtainable from them, were all the naturals and by previous theorem the sum of their reciprocal were finite. In section 3. though we proved it to be infinite.

2.) Let $p_1 = 2$, $p_2 = 3$, . . . $p_n = p$ and then the previous theorems reciprocal and exchanged two sides gives: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) \ldots \left(1-\frac{1}{p}\right) \leq \dfrac{1}{1+\frac{1}{d_1}+\frac{1}{d_2}+\ldots+\frac{1}{d_N}}$

where d_1 , d_2 , . . . d_N are all the numbers obtainable from the primes up to p . Clearly up to p all numbers are obtainable and leaving out all other reciprocals in the right side, the denominator will decrease and the whole right increase so:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) \ldots \left(1-\frac{1}{p}\right) < \frac{1}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{p}}$$

As p increases, the right side denominator becomes arbitrary big so its reciprocal arbitrary small, forcing the left too.

3.)

Trivial by 2.) and last section's theorem used for the $\prod(1-\frac{1}{p})$ associated product of $\sum\frac{1}{p}$.

We give a second proof that repeats the necessary steps of the associated products and thus only relies on 2.) :

First we show that $\frac{1}{p_0} + \frac{1}{p_0*} + \ . \ . \ . + \frac{1}{p} \quad \geq \quad 1 - \left(1-\frac{1}{p_0}\right)\left(1-\frac{1}{p_0*}\right) \ . \ . \ . \ \left(1-\frac{1}{p}\right)$

Where star denotes the next prime and so we have all primes from p_0 to p.

We'll use induction. Equality is obvious for $p = p_0$ Suppose that the equivalent general

$\left(1-\frac{1}{p_0}\right)\left(1-\frac{1}{p_0*}\right) \ . \ . \ . \ \left(1-\frac{1}{p}\right) \quad \geq \quad 1 - \frac{1}{p_0} - \frac{1}{p_0*} - \ . \ . \ . - \frac{1}{p}$ is true up to p

and lets multiply both sides with $\left(1-\frac{1}{p*}\right)$ to get the inequality up to $p*$:

$$\left(1-\frac{1}{p_0}\right)\ldots\left(1-\frac{1}{p*}\right) \geq 1 - \frac{1}{p_0} - \ldots - \frac{1}{p*} + \frac{1}{p*}\left(\frac{1}{p_0}+\ldots+\frac{1}{p}\right) > 1 - \frac{1}{p_0} - \ldots - \frac{1}{p^*}$$

Since by 2.) $\left(1-\frac{1}{2}\right) \ . \ . \ . \ \left(1-\frac{1}{p}\right) \to 0$ but up to a P prime $\left(1-\frac{1}{2}\right) \ . \ . \ . \ \left(1-\frac{1}{P}\right)$ is a finite value, thus starting the product after this, that is $\left(1-\frac{1}{P^*}\right) . . \left(1-\frac{1}{p}\right)$ still tends to 0. In particular for big enough p value this product is always smaller than $\frac{1}{2}$, no matter how big P^* we start with. Then this can be repeated and after the obvious $\left(1-\frac{1}{2}\right) = \frac{1}{2}$ case:

$\left(1-\frac{1}{3}\right) \ . \ . \ . \ \left(1-\frac{1}{p}\right) \quad < \frac{1}{2}$ for $p = P_1$ namely $P_1 = 7$ that is:

$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) \quad < \frac{1}{2}$ indeed $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7} = \frac{48}{105} < \frac{1}{2}$ Then from $7* = 11$:

$\left(1-\frac{1}{11}\right) \ . \ . \ . \ \left(1-\frac{1}{p}\right) \quad < \frac{1}{2}$ for $p = P_2$ but this is a big prime! Then:

$\left(1-\frac{1}{P_2*}\right) \ . \ . \ . \ \left(1-\frac{1}{p}\right) < \frac{1}{2}$ for $p = P_3$ and so on.

Then by our inductively proved inequality:

$$\underbrace{\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}}_{\geq\ 1-\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\ >\ \frac{1}{2}} + \underbrace{\frac{1}{11} + \ . \ . \ . \ . + \frac{1}{P_2}}_{\geq\ 1-\left(1-\frac{1}{11}\right)\ldots\left(1-\frac{1}{P_2}\right)\ >\ \frac{1}{2}} + \underbrace{\frac{1}{P_2^*} + \ . \ . \ . \ . + \frac{1}{P_3}}_{\geq\ 1-\left(1-\frac{1}{P_2*}\right)\ldots\left(1-\frac{1}{P_3}\right)\ >\ \frac{1}{2}} + \ . \ . \ .$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ . \ . \ . \ . = \infty$$

We give a third proof that doesn't even use 2.) .

Suppose $\frac{1}{2}+\frac{1}{3}+\ .\ .\ +\ \frac{1}{p}\ +\ .\ .\ =\ S$ finite. Then up to a p, the sum must be bigger than $S-\frac{1}{2}$ and so from then on the total less than $\frac{1}{2}$, that is: $\frac{1}{p^*}+\frac{1}{p^{**}}+\ .\ .\ .\ <\ \frac{1}{2}$.

So:

$$\begin{array}{ccccccccc} \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \ldots & = & 1 \\ \vee & & \vee & & \vee & & & & \\ \left(\frac{1}{p^*}+\frac{1}{p^{**}}+\ldots\right) & + & \left(\frac{1}{p^*}+\frac{1}{p^{**}}+\ldots\right)^2 & + & \left(\frac{1}{p^*}+\frac{1}{p^{**}}+\ldots\right)^3 & + & \ldots & < & 1 \end{array}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\geq$$

sum of reciprocals with bigger than p prime factors

$$\vee$$

$$\frac{1}{(2\cdot3\cdot\ldots\cdot p)+1}+\frac{1}{2\,(2\cdot3\cdot\ldots\cdot p)+1}+\frac{1}{3\,(2\cdot3\cdot\ldots\cdot p)+1}+\ \ldots\ <\ 1$$

$$\frac{1}{2\cdot3\cdot\ldots\cdot p}\left(\frac{1}{1+\frac{1}{2\cdot3\ldots\ldots p}}+\frac{1}{2+\frac{1}{2\cdot3\ldots\ldots p}}+\frac{1}{3+\frac{1}{2\cdot3\ldots\ldots p}}+\ \ldots\ldots\right)\ <\ 1$$

$$\begin{array}{ccc} \vee & \vee & \vee \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{array}$$

So: $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ .\ .\ .\ .\ <\ 2\cdot3\cdot\ .\ .\ .\ p$ contradicting that the left side is ∞ .

We give a fourth proof.

Again we start from the assumption: $\frac{1}{p^*}+\frac{1}{p^{**}}+\ .\ .\ .\ <\ \frac{1}{2}$.

The number of naturals up to an N that are dividable by a P prime is at most $\frac{N}{P}$ so the number of those that are dividable by at least one of p*, p**, . . . is at most

$\frac{N}{p^*}+\frac{N}{p^{**}}+\ .\ .\ .\ <\frac{N}{2}$

The rest of the naturals up to N , that is those that are not dividable by any of p*, p**, . . . are only dividable by primes up to p. By the above inequality their number is at <u>least</u> $N-\frac{N}{2}=\frac{N}{2}$.

We show that for big enough N, this is impossible. Indeed lets split any n number into its biggest square divider and the remaining prime factors: $n=a^2\cdot2^{b_1}\cdot3^{b_2}\cdot\ .\ .\ .\ \cdot p^{b_k}$ where $b_1, b_2, \ldots b_k$ are all 0 or 1 . Since $a^2 \le n \le N$ thus a can have at most $\sqrt{N}$ values and the b exponents give 2^k many combinations.

So the number of possible naturals with 2 , 3 , . . . p prime factors is at <u>most</u> $\sqrt{N}\cdot 2^k$.

Thus $\frac{N}{2}\le\sqrt{N}\cdot 2^k$ from which: $\sqrt{N}\ \le\ 2^{k+1}$ and thus $N\ \le\ 2^{2k+2}$.

6. Balanced finiteness

R A hidden problem was left unmentioned at the beginning of the previous section when we didn't specify in what order we form the $d_1, d_2, \ldots$ numbers from the $p_1, p_2, \ldots\ p_n$ primes.
We might say, it doesn't matter because the summation doesn't depend on the order since $a + b = b + a$. But here we are dealing with infinite many terms so we could argue that infinite many re-orderings might change the B_n beginning sums so that their limits change, maybe even become infinite. This section deals with exactly such problems but these only appear if we have negative terms too. Indeed a simple logic solves the sums with non-negative terms as follows:
If $x_1 + x_2 + \ldots + x_n = B$ then the re-arranged $x_1' + x_2' + \ldots$ version of the sum will have a beginning that also has at least B value because all the $x_1, x_2, \ldots x_n$ terms are in a beginning of the re-arranged sum. This at once shows that if the original sum was unbounded that is ∞ , then the re-arranged one is also. In reverse, finite sum can only be re-arranged to finite. But we can go a bit further too. If $x_1 + x_2 + \ldots = S$ then any $S - \varepsilon$ value can be surpassed by an $x_1 + x_2 + \ldots + x_n$ beginning, so a re-arrangement must be at least this big, that is the re-arranged S' sum $\geq S - \varepsilon$. But the same goes in reverse from S' to S , so $S \geq S' - \varepsilon$. Then of course only $S = S'$ is possible.

T $$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

1.) < 1
2.) is monotone increasing
3.) has an L limit
4.) $L = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \ldots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

P

1.) $\frac{1}{n+1} + \ldots + \frac{1}{2n} < \frac{1}{n} + \ldots + \frac{1}{n} = 1$

2.) $\left(\frac{1}{n+2} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \ldots + \frac{1}{2n}\right) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} >$
$\frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = 0$

3.) Trivial by 1.) and 2.)

4.) $$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(2n-1)\,2n} =$$

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \quad + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) =$$

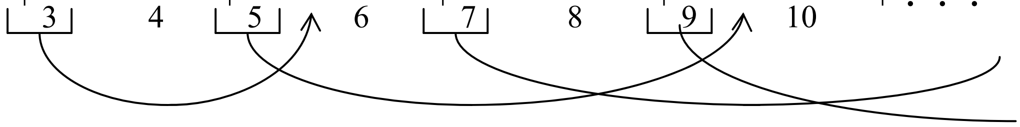
$$\left(1 + \frac{1}{2} - 1\right) + \left(\frac{1}{3} + \frac{1}{4} - \frac{1}{2}\right) + \quad + \left(\frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n}\right) =$$

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

T

If we move in the $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots = L$ sum every odd denominator term further just before the double denominator, then the sum becomes half:

$$1 \quad -\frac{1}{2} \quad +\frac{1}{3} \quad -\frac{1}{4} \quad +\frac{1}{5} \quad -\frac{1}{6} \quad +\frac{1}{7} \quad -\frac{1}{8} \quad +\frac{1}{9} \quad -\frac{1}{10} \quad + \ldots =$$



$$1 \quad -\frac{1}{2} \quad -\frac{1}{4} \quad +\frac{1}{3} \quad -\frac{1}{6} \quad -\frac{1}{8} \quad +\frac{1}{5} \quad -\frac{1}{10} \quad -\frac{1}{12} \quad +\frac{1}{7} \quad - \ldots = \frac{L}{2}$$

P

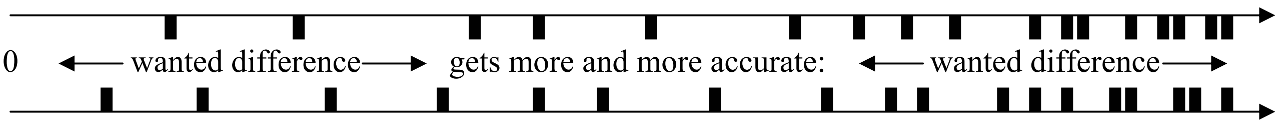
$$\underbrace{1 \quad -\frac{1}{2}} \quad -\frac{1}{4} \quad \underbrace{+\frac{1}{3} \quad -\frac{1}{6}} \quad -\frac{1}{8} \quad \underbrace{+\frac{1}{5} \quad -\frac{1}{10}} \quad -\frac{1}{12} \quad \underbrace{+\frac{1}{7} \quad -\frac{1}{14}} \ldots$$

$$\frac{1}{2} \quad -\frac{1}{4} \quad +\frac{1}{6} \quad -\frac{1}{8} \quad +\frac{1}{10} \quad -\frac{1}{12} \quad +\frac{1}{14} \quad \ldots =$$

$$= \frac{1}{2}\left(1 \quad -\frac{1}{2} \quad +\frac{1}{3} \quad -\frac{1}{4} \quad +\frac{1}{5} \quad -\frac{1}{6} \quad +\frac{1}{7} \ldots\right)$$

R

How could this happen we may ask but there is no reason for alarm.
It was merely a consequence of the anti-Achilles paradox. The positive odd and negative even reciprocals are both slowly diverging sums. Taking any two such and regarding their difference, it can be kept finite even while the two sums increase to infinity. In fact since the sums increase slowly, we can approach with their difference any value we wish:



This same principle in a single sum with $\pm$ terms, means the possibility of shifting the $+$ and $-$ terms relative to each other to get any desired sum value:

T

If a sum contains $a_1, a_2, \ldots$ positive and $-b_1, -b_2, \ldots$ negative terms which both slowly add up to infinity, that is: $a_n \to 0$, $b_n \to 0$ but $\sum a_n = \infty$, $\sum b_n = \infty$ then for any D desired value: $a_1 + \ldots + a_k - b_1 - \ldots - b_m + a_{k+1} \ldots - b_{m+1} \ldots\ldots = D$
Where the order of a-s and b-s in themselves can be the original, only their relative order must be changed.

P

Let's bring to the beginning as many $a_1 + \ldots + a_k$ is required to go above D.
In other words: $a_1 + \ldots + a_{k-1} \le D < a_1 + \ldots + a_k$
Then lets bring after these as many b is required to get under D. Then again with a-s going above then with b-s under D. And so on we clearly alternate around D.
The goings above and under D are guaranteed by the unbounded sums of a-s and b-s .
The approach of D is guaranteed by the slow infiniteness of these sums.
Our proof started with shifting a-s . If D is negative we have to start with b-s .

R

Observe that the changing of sequence in the theorem previous to the above, was not a shift like we used in the above. Even if instead of moving the odd terms back we had brought the doubles forward, it still wouldn't have been a shift because the even terms would have had to jump through other even terms. The last theorem of course means that the same $\frac{L}{2}$ sum value could be achieved by keeping the orders in themselves. Such shifts of course would depend upon the value of L . By the way, soon we'll see what L is actually.

7. The number e Binomials

T

1.) Binomial coefficients as Pascal triangle

$$(a+b)^1 = 1a + 1b$$
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n$$

with $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

In other words every new coefficient is the sum of the previous two above.
Just the coefficient arranged similarly are called the Pascal triangle:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ \cdot & & \cdot & \cdot & & \cdot & \cdot \end{array}$$

2.) The coefficients explicitly

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{2\cdot 3\cdot\ldots\cdot k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} \qquad \text{for } k \geq 1 .$$

Since $\binom{n}{0} = 1$ thus if we accept that $0\,! = 1$ then this form is correct for all k.

3.) Reciprocal binomial

$$\left(1+\frac{x}{n}\right)^n =$$

$$1 + x + \frac{1-\frac{1}{n}}{2} x^2 + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{2\cdot 3} x^3 + \ldots + \frac{\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)}{k!} x^k + \ldots$$

4.) The number **e**

a.) If $x > 0$ then $\left(1+\frac{x}{n}\right)^n$ is monotone increasing from $\left(1+\frac{x}{1}\right)^1 = 1 + x$.

b.) $\left(1+\frac{1}{n}\right)^n$ remains under 3 and

c.) approaches a value between 2 and 3, denoted by **e** .

5.) Identity of monotone double limits

a.) If $P_1 \leq P_2 \leq P_3 \leq \,.\,.\,.\,.\, \to P$ then all $P_n \leq P$ (Similarly with $\geq$)

b.) If $P_n \leq Q_n$ then $\lim P_n \leq \lim Q_n$ or more precisely:
If the $P_1, P_2, \,.\,.\,.\,$ points on the line approach P, $Q_1, Q_2, \,.\,.\,.\,$ approach Q and $P_1 \leq Q_1$, $P_2 \leq Q_2$, $.\,.\,.$ then $P \leq Q$ (Similarly with $\geq$)

c.) If:

$$\begin{array}{ccccccccccc}
B_1^1 & \leq & B_2^1 & \leq & \ldots & \leq & B_k^1 & \leq & \ldots & \to & S_1 \\
B_1^2 & \leq & B_2^2 & \leq & \ldots & \leq & B_k^2 & \leq & \ldots & \to & S_2 \\
\cdot & & \cdot & & & & \cdot & & & & \cdot \\
\cdot & & \cdot & & & & \cdot & & & & \cdot \\
B_1^n & \leq & B_2^n & \leq & \ldots & \leq & B_k^n & \leq & \ldots & \to & S_n \\
\cdot & & \cdot & & & & \cdot & & & & \cdot \\
\cdot & & \cdot & & & & \cdot & & & & \downarrow \\
\downarrow & & \downarrow & & & & \downarrow & & & & S \\
B_1 & & B_2 & & & & B_k & & \ldots & \to & B
\end{array}$$

then $B \leq S$ (Similarly with $\geq$)

d.) If:

$$\begin{array}{ccccccc}
B_1^1 & B_2^1 & \ldots & B_k^1 & \ldots & \to & S_1 \\
/\!\!\wedge & /\!\!\wedge & & /\!\!\wedge & & & \\
B_1^2 & B_2^2 & \ldots & B_k^2 & \ldots & \to & S_2 \\
/\!\!\wedge & /\!\!\wedge & & /\!\!\wedge & & & \\
\cdot & \cdot & & \cdot & & & \cdot \\
B_1^n & B_2^n & \ldots & B_k^n & \ldots & \to & S_n \\
\cdot & \cdot & & \cdot & & & \cdot \\
\cdot & \cdot & & \cdot & & & \downarrow \\
\downarrow & \downarrow & & \downarrow & & & S \\
B_1 & B_2 & & B_k & \ldots & \to & B
\end{array}$$

then $S \leq B$

e.) If

$$\begin{array}{ccccccc}
B_1^1 & \leq & B_2^1 & \leq & \ldots & \to & S_1 \\
/\!\!\wedge & & /\!\!\wedge & & & & \\
B_1^1 & \leq & B_2^1 & \leq & \ldots & \to & S_1 \\
/\!\!\wedge & & /\!\!\wedge & & & & \\
\cdot & & \cdot & & & & \\
 & & & & & & \downarrow \\
\downarrow & & \downarrow & & & & S \\
B_1 & & B_2 & & \ldots & \to & B
\end{array}$$

then $S = B$

f.) If

$$\begin{array}{ccccccccccccc}
t_1^1 & + & t_2^1 & + & \ldots & + & t_k^1 & + & \ldots & = & S_1 \\
\leq & & \leq & & & & \leq & & & & \\
t_1^2 & + & t_2^2 & + & \ldots & + & t_k^2 & + & \ldots & = & S_2 \\
\cdot & & \cdot & & & & \cdot & & & & \cdot \\
t_1^n & + & t_2^n & + & \ldots & + & t_k^n & + & \ldots & = & S_n \\
\cdot & & \cdot & & & & \cdot & & & & \cdot \\
 & & & & & & & & & & \downarrow \\
\downarrow & & \downarrow & & & & \downarrow & & & & S \\
t_1 & + & t_2 & + & \ldots & + & t_k & + & \ldots & = & B
\end{array}$$

where the t-s are all non negative terms, then $S = B$

6.) **e** from the factorial reciprocals

$$e = 2 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots\ldots = \text{irrational}$$

P

1.)
Every new line can be obtained from the previous by multiplying it with (a + b).
Every term in the new line can only come from the two neighboring above, namely the first multiplied by b the second by a.
2.)
For $n = 1$ it can be checked. Assuming it's true up to n and using 1.)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)(n-2)\ldots(n-k+1)}{2\cdot\ldots\cdot(k-1)} + \frac{(n-1)(n-2)\ldots(n-k)}{2\cdot\ldots\cdot k} =$$

$$\frac{[(n-1)(n-2)\ldots(n-k+1)][k+(n-k)]}{2\cdot\ldots\cdot k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{2\cdot\ldots\cdot k}$$

We can give a direct proof without using 1.) by counting how many $a^{n-k}\,b^k$ terms are produced.

$$(a+b)^n = (a+b)\,(a+b)\ldots\ldots(a+b)$$
$$a^{n-k}\,b^k = a\,a\ldots a\,b\,b\ldots b$$

The trick is just to regard the choices of one of the a , b factors, say b-s because then the a-s must come from all the rest of the (a + b)-s . The first b in b^k can come from any of the (a + b) factors and thus has n possibilities. Then the second only from (n – 1), the third from (n – 2) and so on the k-th b from (n – k + 1). This obviously gives $n\,(n-1)\ldots(n-k+1)$ many possibilities, but it contains repeated choices. For example for b^2 , $n\,(n-1)$ counts every b twice, because the second b was already counted as a choice of the first. So only $\frac{n\,(n-1)}{2}$ many b^2-s are. Similarly when more b-s are chosen then even more repeatings are, namely $2\cdot 3\cdot\ldots\cdot k$ at b^k.

3.)

$$\left(1+\frac{x}{n}\right)^n = \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} \frac{x}{n} + \binom{n}{2} 1^{n-2} \frac{x^2}{n^2} + \ldots + \binom{n}{k} 1^{n-k} \frac{x^k}{n^k} + \ldots =$$

$$1 + n\frac{x}{n} + \frac{n(n-1)}{2} \frac{x^2}{n^2} + \ldots\ldots + \frac{n(n-1)..(n-k+1)}{k!} \frac{x^k}{n^k} + \ldots =$$

$$1 + x + \frac{n(n-1)}{n \cdot n} \frac{x^2}{2} + \ldots\ldots + \frac{n(n-1)..(n-(k-1))}{n \cdot n \cdot \ldots \cdot n} \frac{x^k}{k!} + \ldots =$$

$$1 + x + \frac{1-\frac{1}{n}}{2} x^2 + \ldots\ldots + \frac{\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)}{k!} x^k + \ldots$$

The terms become 0 after $k \geq n+1$ so we can regard this as an infinite sum formula. In fact it's useful to write out the $x = 1, n = 1, 2, 3, \ldots$ cases with 0 terms for later use:

$$\left(1+\frac{1}{1}\right)^1 = 2 + 0 + 0 + 0 + \ldots$$

$$\left(1+\frac{1}{2}\right)^2 = 2 + \frac{1-\frac{1}{2}}{2} + 0 + 0 + \ldots$$

$$\left(1+\frac{1}{3}\right)^3 = 2 + \frac{1-\frac{1}{3}}{2} + \frac{\left(1-\frac{1}{3}\right)\left(1-\frac{2}{3}\right)}{2\cdot 3} + 0 + \ldots$$

$$\left(1+\frac{1}{4}\right)^4 = 2 + \frac{1-\frac{1}{4}}{2} + \frac{\left(1-\frac{1}{4}\right)\left(1-\frac{2}{4}\right)}{2\cdot 3} + \frac{\left(1-\frac{1}{4}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{3}{4}\right)}{2\cdot 3\cdot 4} + \ldots$$

4.)

a.) Let $n < N$. Then:

$$1 + x + \frac{1-\frac{1}{n}}{2} x^2 + \ldots + \frac{\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)}{k!} x^k + \ldots + \frac{\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)}{n!} x^n <$$

$$\wedge \qquad\qquad \wedge$$

$$1 + x + \frac{1-\frac{1}{N}}{2} x^2 + \ldots + \frac{\left(1-\frac{1}{N}\right)\ldots\left(1-\frac{k-1}{N}\right)}{k!} x^k + \ldots + \frac{\left(1-\frac{1}{N}\right)\ldots\left(1-\frac{N-1}{N}\right)}{N!} x^N$$

b.)

$$\left(1+\frac{1}{n}\right)^n = 2 + \frac{1-\frac{1}{n}}{2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{2\cdot 3} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{2\cdot 3\cdot 4} + \ldots$$

$$\wedge \qquad \wedge \qquad \wedge$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} < 1$$

c.) Follows from a.) and b.)

5.)

a.) If a $P_N > P$ were, then all P_n after P_N were further away from P than P_N and thus couldn't approach P

b.) If $P > Q$ were, then after a P_N all P_n were larger than the middle point of P , Q Then of course all Q_n were too and thus couldn't approach Q

c.) By a.) $B_k^n \leq S_n$ so by b.) $B_k \leq S$ and so again by b.) $B \leq S$

d.) By a.) $B_k^n \leq B_k$ so by b.) $S_n \leq B$ and so again by b.) $S \leq B$

e.) Follows from b.) and c.)

f.) Follows from e.) with $B_k^n = t_1^n + t_2^n + \ldots t_k^n$

6.)

Follows from 5.) f.) as:

$$\left(1+\frac{1}{1}\right)^1 = 2 + 0 + 0 + 0 + \ldots$$

$$\left(1+\frac{1}{2}\right)^2 = 2 + \frac{1-\frac{1}{2}}{2} + 0 + 0 + \ldots$$

$$\left(1+\frac{1}{3}\right)^3 = 2 + \frac{1-\frac{1}{3}}{2} + \frac{\left(1-\frac{1}{3}\right)\left(1-\frac{2}{3}\right)}{2\cdot 3} + 0 + \ldots$$

$$\left(1+\frac{1}{4}\right)^4 = 2 + \frac{1-\frac{1}{4}}{2} + \frac{\left(1-\frac{1}{4}\right)\left(1-\frac{2}{4}\right)}{2\cdot 3} + \frac{\left(1-\frac{1}{4}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{3}{4}\right)}{2\cdot 3\cdot 4} + \ldots$$

$$\vdots \qquad \downarrow$$

$$\mathbf{e} = 2 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots$$

The irrationality follows from the one proven in section 2. Squares and factorials.

R The identical monotony of the rows and columns is necessary as the following shows:

$$\begin{array}{cccccccccccccccc}
1 & , & 1 & , & 1 & , & 1 & , & 1 & , & 1 & \ldots\ldots & \rightarrow & 1 \\
\frac{1}{2} & , & 1 & , & 1 & , & 1 & , & 1 & , & 1 & \ldots\ldots & \rightarrow & 1 \\
\frac{1}{3} & , & \frac{2}{3} & , & 1 & , & 1 & , & 1 & , & 1 & \ldots\ldots & \rightarrow & 1 \\
\frac{1}{4} & , & \frac{2}{4} & , & \frac{3}{4} & , & 1 & , & 1 & , & 1 & \ldots\ldots & \rightarrow & 1 \\
\frac{1}{5} & , & \frac{2}{5} & , & \frac{3}{5} & , & \frac{4}{5} & , & 1 & , & 1 & \ldots\ldots & \rightarrow & 1 \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & & \cdot \\
 & & & & & & & & & & & & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & 1 \\
0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 & \ldots\ldots & \rightarrow & 0
\end{array}$$

8. Rational exponents

T The $x^n = x\,x \ldots x$ power is monotone increasing if the non negative x is.
This also means that x^n is unique for non negative x values.

P Let two values of x be $x < x + d$. Then: $(x+d)^n = x^n + (n\,x^{n-1}d + \ldots + d^n) > x^n$ because $(\ldots)$ is positive.

D 1.)
If z is any number then $\sqrt[n]{z}$ denotes the collection of all such y numbers that $y^n = z$.
For $\sqrt[2]{}$ we simply use $\sqrt{}$. For example $\sqrt{4} = \{2, -2\}$.
Later, this collection of "roots" or "surds" will be more interesting for $z =$ plane numbers.
2.)
By the previous theorem if we restrict ourselves to non negative reals, then the surd is unique. In fact it can be regarded as a new operation. So if for the x non negative real $\sqrt[n]{x}$ denotes the unique element itself, then we introduce reciprocal exponents as: $x^{\frac{1}{n}} := \sqrt[n]{x}$
3.)
By the well known rule: $\left(x^n\right)^k = (x\ldots x)(x\ldots x)\ldots\ldots(x\ldots x) = x \ldots\ldots x = x^{n\,k}$
we can extend the reciprocal exponents to any fractions:

$$x^{\frac{k}{n}} = x^{\frac{1}{n}k} := \left(x^{\frac{1}{n}}\right)^k = \left(\sqrt[n]{x}\right)^k$$

4.)
With similar "tricky" arguments we can guess the proper definitions to extend the exponents for 0 and negative fractions. Indeed if $n > k$ then $x^{n-k} = \dfrac{x^n}{x^k} = \dfrac{x.\ldots.x}{x\ldots x} = x..x$.

So then with $n = k$ we should have: $x^0 = x^{n-n} = \dfrac{x^n}{x^n} = 1$ and then for negative wholes:

$$x^{-n} = x^{0-n} = \frac{x^0}{x^n} = \frac{1}{x^n} \qquad \text{and then in general:} \qquad x^{-\frac{k}{n}} := \frac{1}{x^{\frac{k}{n}}} = \frac{1}{\left(\sqrt[n]{x}\right)^k}$$

R Now it's quite easy to prove that our "tricky" definitions "worked", that is the well known five rules of powers remain true for rational r, s exponents. The bases x, y must be of course non negative reals, in fact for two of the rules to be positive:

1.) $(x\,y)^r = x^r\,y^r$ 2.) $\left(\dfrac{x}{y}\right)^r = \dfrac{x^r}{y^r}$ $(y \neq 0)$

3.) $x^{r+s} = x^r\,x^s$ 4.) $x^{r-s} = \dfrac{x^r}{x^s}$ $(x \neq 0)$

5.) $\left(x^r\right)^s = x^{r\,s} = x^{s\,r} = \left(x^s\right)^r$

These and the monotony of the powers give tools to generalize many earlier results. Remember that monotony not only means that the power increases with the base at positive exponent but that an inequality can be raised to positive exponents.

T

1.) If $1 < \frac{A}{a} < \frac{B}{b}$ fractions then $\left(1+\frac{1}{\frac{A}{a}}\right)^{\frac{A}{a}} < \left(1+\frac{1}{\frac{B}{b}}\right)^{\frac{B}{b}}$. In other words:

$\left(1+\frac{1}{R}\right)^{R}$ is monotone increasing with R, for $R > 1$ rationals.

2.) If $r > 0$ rational then: $\left(1+\frac{r}{n}\right)^{\frac{n}{r}} \to e$

3.) If $r > 0$ rational then: $\left(1+\frac{r}{n}\right)^{n} \to e^{r}$

4.) If $r > 0$ rational then: $(1+r) < e^{r}$

P

1.) We already proved in the last section that if $n < N$ then: $\left(1+\frac{x}{n}\right)^{n} < \left(1+\frac{x}{N}\right)^{N}$

Lets use this with $n = A\,b$, $N = B\,a$, $x = a\,b$ and take $(a\,b)$ root of both sides.

2.) Let $n > r$ and then by 1.)

$$\left(1+\frac{1}{\left[\frac{n}{r}\right]}\right)^{\left[\frac{n}{r}\right]} \le \left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}} \le \left(1+\frac{1}{\left[\frac{n}{r}\right]+1}\right)^{\left[\frac{n}{r}\right]+1}$$

The two sequences on the left and right are subsets of $\left(1+\frac{1}{n}\right)^{n}$ so they approach **e** , forcing the in between one also.

3.) By 2.) $\left(1+\frac{r}{n}\right)^{n} = \left[\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^{r} \to e^{r}$

4.) By the proved monotony and 3.) : $(1+r) = \left(1+\frac{r}{1}\right)^{1} < \left(1+\frac{r}{2}\right)^{2} < \ldots \to e^{r}$

Thus all elements are smaller than e^{r} including $(1+r)$.

R

The rational exponents give a formalism that makes many proofs much shorter. We have to realize that these are illusions because if the assumed rules were to be proven then we would end up with lot more steps. Still I show a “nice” example:

T

If $r_1 , r_2 , \ldots > 0$ then: $\sum r = \infty \iff \prod (1-r) = \infty$

P

$1 + r_1 + r_2 + \ldots < (1+r_1)\,(1+r_2) \ldots < e^{r_1}\, e^{r_2} \ldots = e^{(r_1 + r_2 + \ldots)}$

So if $r_1 + r_2 + \ldots = \infty$, then this forces $(1+r_1)\,(1+r_2) \ldots$ and then this $e^{(r_1 + r_2 + \ldots)}$ and that $r_1 + r_2 + \ldots$ to be ∞

R Now I give an example where the new proof is not even shorter, in fact much inferior too.

We had already four proofs for the infinity of the prime reciprocal sum, so here comes a fifth. The inferiority of it, is not that it uses rational exponents, neither that it is longer, but that it uses the unique prime factorizations. In fact it uses this for the square numbers and then later the prime factorization without uniqueness for all numbers. The previous four proofs used only this non unique prime factorization, which is trivial because every number can be written as a prime product by dividing it until it's possible. On the other hand the uniqueness is far from obvious.

T $$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ldots+\frac{1}{p}+\ldots=\infty$$

P Suppose the sum is an S finite value. Then: $e^S = e^{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots} =$

$$e^{\frac{1}{2}} \cdot e^{\frac{1}{3}} \cdot e^{\frac{1}{5}} \cdot \ldots < e^S$$

$$\vee \qquad \vee \qquad \vee$$

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\ldots < e^S$$

Since $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots<2$, thus

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\ldots\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\right) < e^S\cdot 2$$

By the unique prime factorization of squares:

$$\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\ldots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\ldots\right)\ldots$$

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\ldots\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots\right)\ldots < e^S\cdot 2$$

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\ldots\right)\ldots < e^S\cdot 2$$

By the prime factorization of all numbers: $\backslash\!\vee$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots < e^S\cdot 2$$

It's a clear contradiction because the left is ∞ .

R Finally I would like to mention that the rational exponents can be easily generalized to any real numbers as the limits of powers with rational exponents approaching the real. Of course the uniqueness of this must be proven. Later we examine much more closely how the rationals lie among all real numbers.

9. Compound limit

R

Numbers are the ratios of distances but in a "Banker's Geometry" they are ratios of money.
Even more important is the ratios of profits that is the rate of increase on every dollar. If this ratio is r then 1 dollar becomes 1 + r and a P principle P (1 + r) . The usual interest rates can be easily translated into this.
For example: 5 % means r = .05 , 13 % means r = .13 , 13.5 % means r = .135
and of course 100 % means r = 1. In general: R % means $r = \frac{R}{100}$.
Of course, all interest rates are relative to a time, usually to a year. If someone leaves his money only for half year he only gets $\left(1+\frac{r}{2}\right)$ times his money back. Clearly if somebody takes his money out after half year and then puts it back again, then at the end of the year, he would have: $\left(1+\frac{r}{2}\right)\left(1+\frac{r}{2}\right)=\left(1+r+\frac{r^2}{4}\right)$ times his original money. Now if somebody just leaves the same money all year he gets (1 + r) times which is $\frac{r^2}{4}$ less. This is clearly not fair and therefore the banks automatically add the interest to our money without us have to do the trick of taking it out and putting it back. But how often they should do this so called "compounding" ? In our example it was only once in the middle of the year, and that meant $\frac{r^2}{4}$ ratio gain already. Maybe if we do it more and more frequently we can even increase our money without limit. This is not so! With monthly compounding the total ratio is $\left(1+\frac{r}{12}\right)^{12}$, with weekly it is $\left(1+\frac{r}{52}\right)^{52}$ and with daily $\left(1+\frac{r}{365}\right)^{365}$. But as we proved $\left(1+\frac{r}{n}\right)^{n} \to e^{r}$.
This gives a very mundane definition of the number **e** , if we repeat our argument for r = 1:
One dollar with 100 % yearly interest rate increases to 1+1 = 2 dollars.
If we add the partial interests more and more often, then our dollar still can't become infinite, instead it approaches a value, which is: **e** = 2.718 . . .
From the actual value of **e** we also see that about 72 cents is a definite limit of the extra interest that we can gain from compounding. Though it's not infinite as the naive wishful thinking assumed, it is still a considerable gain. However with more realistic smaller interest rates it decreases considerably. Indeed in general at r ratio the uncompounded increased value of a dollar is 1 + r and the infinitely compounded is e^r so the extra interest is: $e^{r}-(1+r)$.
Thus at r = .1 that is R=10% we get $e^{.1}-1.1$ = 1.10517. . – 1.1 = .00517. So our extra interest from the compounding is only half cent. From actual calculations we can also see that at normal interest rates the increased amount hardly changes by increasing the number of compoundings. For example from weekly to daily, the increase would be less than thousandths of a cent. So in practice we can use the theoretical limit, that is e^r because it is easier to calculate. In short every dollar becomes about e^r.

When we take money from the bank the situation is reversed, we have to pay interest and the principle is called an L loan. An other difference is that banks always ask more interest from us then they give for our money, otherwise of course they couldn't make money for themselves. Finally there is a third difference, namely that banks require us to make regular P payments, say with f frequency a year. We can make a sequence of B balances after every payment:

$B_0 = L$ At the time of taking the loan the balance is L , we owe the whole loan.

$B_1 = L\left(1+\frac{r}{f}\right) - P$ after first payment.

$B_2 = B_1\left(1+\frac{r}{f}\right) - P = L\left(1+\frac{r}{f}\right)^2 - P\left(1+\frac{r}{f}\right)$ after second payment. And so on.

.

.

$$B_n = L\left(1+\frac{r}{f}\right)^n - P\left(1+\frac{r}{f}\right)^{n-1} - P\left(1+\frac{r}{f}\right)^{n-2} - \ldots\ldots - P =$$

$$L\left(1+\frac{r}{f}\right)^n - P\left[\left(1+\frac{r}{f}\right)^{n-1} + \left(1+\frac{r}{f}\right)^{n-2} + \ldots + 1\right] = L\left(1+\frac{r}{f}\right)^n - P\,\frac{\left(1+\frac{r}{f}\right)^n - 1}{\left(\cancel{1+}\frac{r}{f}\right)\cancel{-1}}$$

$$\left(1+\frac{r}{f}\right)^n = \left[\left(1+\frac{r}{f}\right)^{\frac{f}{r}}\right]^{\frac{n\,r}{f}} \cong e^{\frac{n\,r}{f}} \qquad \text{So:} \quad B_n \cong L\,e^{\frac{n\,r}{f}} - P\,\frac{f}{r}\,\left(e^{\frac{n\,r}{f}} - 1\right)$$

The loan is paid when $B_n = 0$, that is: $L\,e^{\frac{n\,r}{f}} = P\,\frac{f}{r}\,\left(e^{\frac{n\,r}{f}} - 1\right)$.

We can express L or P : $\quad L = P\,\frac{f}{r}\left(1 - \frac{1}{e^{\frac{n\,r}{f}}}\right)$ and $P = L\,\frac{r}{f}\,\frac{1}{1 - \frac{1}{e^{\frac{n\,r}{f}}}}$

Multiplying this with n : $\quad P\,n = L\,\frac{\frac{n\,r}{f}}{1 - \frac{1}{e^{\frac{n\,r}{f}}}} = L\,\frac{y\,r}{1 - \frac{1}{e^{y\,r}}}$

Where $y = \frac{n}{f}$ is the time in which the loan is repaid measured in years.

The $P\,n$ on the left is the total money we paid "back". As we see, this only depends on $y\,r$ and of course on the L loan. The fraction multiplier can be regarded as the "robbing factor" that is, how much more the bank gets for every dollar that was lent to us.

Many times we are interested in how long it will take to pay off a loan. For this:

From the equation of L : $\quad 1 - \frac{1}{e^{\frac{n\,r}{f}}} = \frac{L\,r}{P\,f}$ from this we can express: $\quad e^{\frac{n\,r}{f}} = \frac{1}{1 - \frac{L\,r}{P\,f}}$

To express the exponent with which a base gives a certain power, we use the so called logarithm. For example: $10^3 = 1000$ so the base 10 logarithm of 1000 is 3 . As we remember, negative exponents meant reciprocal powers, so in reverse the logarithm of a reciprocal is negative.

Using $\ln$ for the base **e** logarithm from the latin "logarithm natural", we get for our case:

$$\frac{n\,r}{f} = -\ln\left(1 - \frac{L\,r}{P\,f}\right) \qquad \text{so} \quad \frac{n}{f} = y = \frac{-\ln\left(1 - \frac{L\,r}{P\,f}\right)}{r}$$

<u>10. Continued reciprocals</u> Lambert's sum ratio formula

D

Let x_1 , x_2 , ... be positive real numbers, then:

1.) We call: $\dfrac{y_1}{x_1}$, $\dfrac{y_1}{x_1+\dfrac{y_2}{x_2}}$, $\dfrac{y_1}{x_1+\dfrac{y_2}{x_2+\dfrac{y_3}{x_3}}}$, ...

a continued fraction sequence (c.f.s.) and its elements as the continued fractions of the sequence.

2.) If y_1 , y_2 , ... are also all positive then we talk about additive c.f.s. , while if they are all negative then about subtractive c.f.s.

3.) If y_1 , y_2 , ... are all ± 1 then we call the c.f.s. as continued reciprocal sequence (c.r.s.) .

4.) Then of course if y_1 , y_2 , ... are all 1 we talk about additive c.r.s. while if all are - 1 then subtractive c.r.s.

T

1.) The two heuristic recursions:

The $F_n = F(\, y_1\ x_1\ .\ .\ .\ .\ y_n\ x_n\,)$ n-th case of a formula for the elements of a c.f.s. can be obtained from the (n – 1)-th case in two ways as:

a.) $F(\, y_1\, x_1\ .\ .\ .\ .\ y_{n-1}\ x_{n-1} := x_{n-1} + \dfrac{y_n}{x_n}) \quad = \quad F_{n-1}(\, x_{n-1} := x_{n-1} + \dfrac{y_n}{x_n})$

b.) $\dfrac{y_1}{x_1 + F\left(y_1 := y_2 , x_1 := x_2 , \ .\ .\ .\ .\ y_{n-1} := y_n , x_{n-1} := x_n \right)} = \dfrac{y_1}{x_1 + (F_{n-1})'}$

2.) Change of c.f.s. to c.r.s.

$$\cfrac{y_1}{x_1+\cfrac{y_2}{x_2+\cfrac{y_3}{x_3+\cdots+\cfrac{y_n}{x_n}}}} =$$

$$= \cfrac{1}{\cfrac{x_1}{y_1}+\cfrac{1}{\cfrac{x_2}{y_2}\,y_1+\cfrac{1}{\cfrac{x_3}{y_3}\,\cfrac{y_2}{y_1}+\cfrac{1}{\cfrac{x_4}{y_4}\,\cfrac{y_3\,y_1}{y_2}+\cdots+\cfrac{1}{\cfrac{x_n}{y_n}\,\cfrac{y_{n-1}\,y_{n-3}\ldots(y_1 \text{ or } y_2)}{y_{n-2}\,y_{n-4}\ldots(y_2 \text{ or } y_1)}}}}}}$$

P

1.) Trivial from the formation of c.f.s.

2.) For the first few members it's easy to check.
Then by our first a.) heuristic rule it follows from the (n – 1)-th to the n-th:

$$\frac{x_{n-1}+\frac{y_n}{x_n}}{y_{n-1}}\;\frac{y_{(n-1)-1}\,y_{(n-1)-3}\cdots}{y_{(n-1)-2}\,y_{(n-1)-4}\cdots} = \frac{x_{n-1}}{y_{n-1}}\;\frac{y_{n-2}\,y_{n-4}\cdots}{y_{n-3}\,y_{n-5}\cdots} + \underbrace{\frac{y_n}{x_n\,y_{n-1}}\;\frac{y_{n-2}\,y_{n-4}\cdots}{y_{n-3}\,y_{n-5}\cdots}}_{\dfrac{1}{\frac{x_n}{y_n}\;\frac{y_{n-1}\,y_{n-3}\cdots}{y_{n-2}\,y_{n-4}\cdots}}}$$

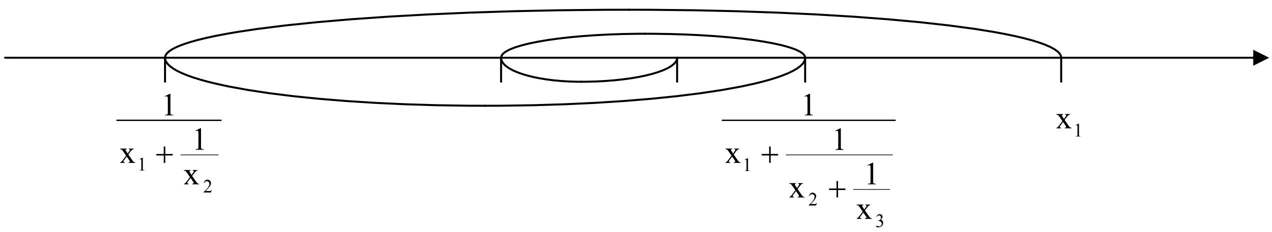
R

The usual terminology calls continued fraction what we called the whole sequence and the elements as “convergents” . The continued reciprocals are referred to as “ordinary” continued fractions. The name convergents is not only stupid but false too because they don't necessarily converge. The question of convergence is fairly complicated in general but for additive and subtractive c.r.s. we have nice results.
The subtractive sequences are much more complicated than the additive ones, not only because they can have both positive and negative values but because some of the earlier denominators can become 0 , thus making the whole continued fraction meaningless.
This explains the huge difference between the conditions we give for the additive and subtractive c.r.s.:

T

1.) Additive alternation

The additive continued reciprocals alternatively decrease and increase but the new values are always in between all old values:



Thus the odd and even members separately always have limits and the question is only whether these two limits are the same, in other words whether the gap between the odd and even members is diminishing to 0.

2.) Additive gap diminishing criteria

If $x_1 + x_2 + \ldots = \infty$ then the gap above $\rightarrow 0$ so there is a single limit of the c.r.s.

3.) Subtractive alternation

If $x_1 , x_2 , \ldots . \geq 2$ then the subtractive continued reciprocals alternate exactly like the additive ones except, all values are negative between 0 and -1.

4.) Subtractive gap diminishing

If $x_1 , x_2 , \ldots . \geq 2$ then the gap automatically diminishes too.

P

1.) $\frac{1}{x_1} > \frac{1}{x_1+\frac{1}{x_2}}\ ,\ \frac{1}{x_1+\frac{1}{x_2+\frac{1}{x_3}}}\ ,\ \ldots$ The same is true with shifted variables:

$\frac{1}{x_2} > \frac{1}{x_2+\frac{1}{x_3}}\ ,\ \frac{1}{x_2+\frac{1}{x_3+\frac{1}{x_4}}}\ ,\ \ldots$ This added in the denominator of $\frac{1}{x_1}$:

$\frac{1}{x_1+\frac{1}{x_2}} < \frac{1}{x_1+\frac{1}{x_2+\frac{1}{x_3}}}\ ,\ \frac{1}{x_1+\frac{1}{x_2+\frac{1}{x_3+\frac{1}{x_4}}}}\ ,\ \ldots .$ Shifted again:

$\frac{1}{x_2+\frac{1}{x_3}} < \ldots .$ Added again in $\frac{1}{x_1}$:

$\frac{1}{x_1+\frac{1}{x_2+\frac{1}{x_3}}} > \ldots .$

And so on we proved both the alternation and in betweeness.

2.)

For this we have to go deeper in to the recursion of c.f.s. formulas.

If we calculate the additive continued reciprocals with general $x_1, x_2, \ldots$ variables then we always end up to a fraction like form:

$$\frac{1}{x_1} = \frac{1}{x_1}$$

$$\frac{1}{x_1+\frac{1}{x_2}} = \frac{x_2}{x_1 x_2 + 1}$$

$$\frac{1}{x_1+\frac{1}{x_2+\frac{1}{x_3}}} = \frac{1}{x_1+\frac{x_3}{x_2 x_3+1}} = \frac{x_2 x_3 + 1}{x_1 x_2 x_3 + x_1 + x_3}$$

$$F_n = \frac{N_n}{D_n}$$

The advantage of this will be to get better rules for N and D separately then for the whole F:

a.) $N_n = x_n N_{n-1} + N_{n-2}$ Similarly: $D_n = x_n D_{n-1} + D_{n-2}$

Indeed we can directly check for $n = 3$ and then if it's true up to $(n - 1)$ then with our first heuristic rule:

$$\frac{N_n}{D_n} = \frac{N_{n-1}\,(x_{n-1} := x_{n-1} + \frac{1}{x_n})}{D_{n-1}\,(\ldots)} = \frac{(x_{n-1}+\frac{1}{x_n})\, N_{n-2} + N_{n-3}}{\text{same with } D} \cdot \frac{x_n}{x_n} =$$

$$\frac{x_n x_{n-1} N_{n-2} + x_n N_{n-3} + N_{n-2}}{\text{same with } D} = \frac{x_n N_{n-1} + N_{n-2}}{\text{same with } D}$$ So the rules stand for n too.

Our new second rule again follows from the old second rule:

b.) $\frac{N_n}{D_n} = \frac{1}{x_1 + \frac{(N_{n-1})'}{(D_{n-1})'}} = \frac{(D_{n-1})'}{x_1 (D_{n-1})' + (N_{n-1})'}$ Apostrophe denotes the shifted variables.

Now we apply these new rules for what we are really interested in that is for the differences of consecutive values. Using b.) without subscripts this difference is:

$$\frac{N}{D} - \frac{D'}{x_1 D' + N'} = \frac{N(x_1 D' + N') - DD'}{D(x_1 D' + N')} = \frac{x_1 ND' + NN' - DD'}{D(x_1 D' + N')}$$

For the next two members the difference is more complicated:

$$\frac{D'}{x_1 D' + N'} - \frac{x_2 D'' + N''}{x_1 (x_2 D'' + N'') + D''} = \frac{D'\left[x_1 (x_2 D'' + N'') + D''\right] - (x_1 D' + N')(x_2 D'' + N'')}{\text{product of two denom}} =$$

$$\frac{x_1 x_2 D'D'' + x_1 D'N'' + D'D'' - x_1 x_2 D'D'' - x_1 D'N'' - x_2 N'D'' - N'N''}{\text{product of two denom}} =$$

$$\frac{-x_2 N'D'' - N'N'' + D'D''}{\text{product}} =$$

$$\frac{-\left[x_1 ND' + NN' - DD'\right]'}{\text{product}} = \frac{-\left[\text{nominator of previous difference}\right]'}{\text{product}}$$

Since the difference for the first two members is $\frac{1}{x_1} - \frac{x_2}{x_1 x_2 + 1} = \frac{1}{x_1 (x_1 x_2 + 1)}$

Thus the numerators of differences start with 1 and by our result keep to be $\pm\ 1$.
Therefore to prove that the differences diminish, all we have to show is that the $D_n D_{n+1}$ product of consecutive denominators is unbounded in value if $x_1 + x_2 + \ldots$ is also.
Since one of D_n , D_{n+1} is always odd and the other even thus it's enough to prove that the separate odd and even denominators D_1 , D_3 , D_5 , . . . and D_2 , D_4 , D_6 , . . . both have positive lower bounds and at least one of them is unbounded if $x_1 + x_2 + \ldots$ is.

We claim that: $D_n > x_1 + x_3 + \ldots + x_n$ for odd n

and $D_n > x_1 (x_2 + x_4 + \ldots + x_n)$ for even n .

To show these we use the new rule a.) for the denominators:

$$x_1 \quad , \quad \lfloor x_1 x_2 + 1 \rfloor \quad , \quad \lfloor x_1 x_2 x_3 + x_1 + x_3 \rfloor \quad , \quad x_4(\ ) + (\ ) \quad , \quad \ldots$$

This at once shows that the even members are all greater than 1 and the odds greater than x_1 and using the rule in general too:

$$D_n = x_n D_{n-1} + D_{n-2} > \begin{cases} x_n \cdot 1 + D_{n-2} & \text{for odd } n. \\ x_n x_1 + D_{n-2} & \text{for even } n. \end{cases}$$

Using the same for D_{n-2} and then for D_{n-4} and so on:

$D_n > x_n + x_{n-2} + x_{n-4} + \ldots$ for odd n.

$D_n > x_n x_1 + x_{n-2} x_1 + x_{n-4} x_1 + \ldots$ for even n.

3.)

If we could guarantee that the denominators are all positive then similar argument as used in 1.) shows the alternation. The $x_1 , x_2 \ldots \geq 2$ condition is a strong over securing of the positiveness because it actually guarantees that the denominators are bigger than 1 .
Indeed, the last denominator in the n-th continued reciprocal is $x_n \geq 2 > 1$ by our conditions.

So: $x_{n-1} - \frac{1}{x_n} \geq 2 - \frac{1}{x_n} > 2 - 1 = 1$ Then $x_{n-2} - \frac{1}{x_{n-1} - \frac{1}{x_n}} > 2 - 1 = 1$ and so on.

4.)
Strangely, a very similar argument as used in 3.) can work for the gap diminishing too: Corresponding formulas as a.) and b.) in 2.) are easy to derive. Again the numerators of differences are ± 1 so the infiniteness of the $D_n D_{n+1}$ denominator is all we need. Here however we don't need to separate the odd and even members because we'll see that they both $\rightarrow \infty$.
The denominator recursion is now: $D_n = x_n D_{n-1} - D_{n-2}$
All we need is $D_n \rightarrow \infty$ but going "one step" beyond we'll show: $D_n \geq D_{n-1} + 1$ and the "two step" beyond $x_n \geq 2$ guarantees this:
Unlike in the alternation above where we went backwards in the denominators, here we simply start at the beginning: $D_1 = x_1$, $D_2 = x_1 x_2 - 1$, . . . So by $x_1, x_2 \geq 2$ we get:
$D_2 = x_1 x_2 - 1 \geq x_1 2 - 1 = x_1 + x_1 - 1 \geq x_1 + 2 - 1 = D_1 + 1$
Suppose it's true up to D_{n-1} then:
$D_n = x_n D_{n-1} - D_{n-2} \geq 2 \cdot D_{n-1} - D_{n-2} = D_{n-1} + D_{n-1} - D_{n-2} \geq D_{n-1} + 1$

R

1.)
As can be shown, the $x_1 + x_2 + \ . \ . \ . = \infty$ condition for the additive case is not only sufficient but necessary too. This means that we have three beautiful equivalent conditions:

$$\sum x = \infty \quad \Leftrightarrow \quad \prod (1 + x) = \infty \quad \Leftrightarrow \quad \cfrac{1}{x_1 + \cfrac{1}{x_2 + \dots}} = \text{finite limit}$$

The $x_1, x_2, \ . \ . \ . \geq 2$ condition for the subtractive case is not a necessary one. In fact for positive $x_1, x_2, \ . \ . \ . < 1$, we can similarly prove the sufficiency. For $x_1, x_2, \ . \ . \ . \ .$ between 1 and 2 , the question of limit is quite complicated. To illustrate this I just mention the followings:
For $x_n = 2 - \frac{1}{n}$ there is limit but for $x_n = 2 - \frac{1}{n^{1+p}}$ with any positive p there isn't.
2.)
If a c.r.s. is known, claimed or assumed to be convergent then usually we just list it as an infinite continued reciprocal nest and equality of these means existence of limits too:

T

Lambert's sum ratio formulas. If x is positive then:

1.)
$$\frac{1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} + \dots}{x + \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} + \dots} = \cfrac{1}{x} + \cfrac{1}{\cfrac{3}{x} + \cfrac{1}{\cfrac{5}{x} + \cfrac{1}{\dots}}}$$

2.)
$$\frac{1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} - \dots}{x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \dots} = \cfrac{1}{x} - \cfrac{1}{\cfrac{3}{x} - \cfrac{1}{\cfrac{5}{x} - \cfrac{1}{\dots}}}$$

P

1.)

The denominator's sum is a higher power version of the numerator's.
We can continue this as:

$x^2 + \frac{x^4}{2\cdot 3\cdot 4} + \frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \;....\;$ Then the next starting with x^3 and so on.

But there are other ways to look at the numerator's and denominator's sum that yield different continuations.
Using the $2\cdot 3\cdot\;....\;\cdot n = n!$ notation and accepting $0! = 1$ we can define the following sequence of sums:

$$\Sigma_0 := 2^0\left[\frac{0!\,x^0}{0!\,(2\cdot 0)!} + \frac{1!\,x^2}{1!\,2!} + \frac{2!\,x^4}{2!\,4!} + \;.......\right]$$

$$\Sigma_1 := 2^1\left[\frac{1!\,x^1}{0!\,(2\cdot 1)!} + \frac{2!\,x^3}{1!\,4!} + \frac{3!\,x^5}{2!\,6!} + \;.......\right]$$

$$\Sigma_2 := 2^2\left[\frac{2!\,x^2}{0!\,(2\cdot 2)!} + \frac{3!\,x^4}{1!\,6!} + \frac{4!\,x^6}{2!\,8!} + \;.......\right]$$

$$\Sigma_3 := 2^3\left[\frac{3!\,x^3}{0!\,(2\cdot 3)!} + \frac{4!\,x^5}{1!\,8!} + \frac{5!\,x^7}{2!\,10!} + \;.......\right]$$

$$\vdots$$

$$\Sigma_n := 2^n\left[\frac{n!\,x^n}{0!\,(2n)!} + \frac{(n+1)!\,x^{n+2}}{1!\,(2n+2)!} + \;.....\; + \frac{(n+k)!\,x^{n+2k}}{k!\,(2n+2k)!} + \;.......\right]$$

As we can check Σ_0 and Σ_1 are our numerator and denominator but what is the advantage of this over complicated way of looking at them? The following:

$$\Sigma_n - \Sigma_{n+2} = \Sigma_{n+1}\cdot\frac{2n+1}{x}$$

The real beauty is that this can be proved term by term as:

$(\Sigma_n)_k - (\Sigma_{n+2})_{k-1} = (\Sigma_{n+1})_k\cdot\frac{2n+1}{x}$ where $(\;)_k$ denotes the k-th term of the sum starting from 0. As we see, we have to subtract the one-earlier term and this means that at $k = 0$ $(\Sigma_n)_0 = (\Sigma_{n+1})_0\cdot\frac{2n+1}{x}$ must stand since there is no -1-th term. And indeed:

$$2^n\cdot\frac{n!\,x^n}{0!\,(2n)!} = 2^{n+1}\cdot\frac{(n+1)!\,x^{n+1}}{0!\,(2n+2)!}\cdot\frac{2n+1}{x} = 2^n\cdot\cancel{2}\cdot\frac{n!\,\cancel{(n+1)}\,x^n\,\cancel{x}}{0!\,(2n)!\,\cancel{(2n+1)}\,\cancel{(2n+2)}}\cdot\frac{\cancel{2n+1}}{\cancel{x}}$$

The previous general k-th term relation can be checked similarly with more calculation.

From the thus proved $\Sigma_n - \Sigma_{n+2} = \Sigma_{n+1}\cdot\frac{2n+1}{x}$ relation we get:

$\frac{\Sigma_n}{\Sigma_{n+1}} = \frac{2n+1}{x} + \frac{\Sigma_{n+2}}{\Sigma_{n+1}} = \frac{2n+1}{x} + \cfrac{1}{\cfrac{\Sigma_{n+1}}{\Sigma_{n+2}}}$ and here for $\frac{\Sigma_{n+1}}{\Sigma_{n+2}}$ we can apply again the $\frac{\Sigma_n}{\Sigma_{n+1}}$ formula itself and so starting from the beginning, we get:

$$\frac{\Sigma_0}{\Sigma_1} = \frac{2\cdot 0+1}{x} + \cfrac{1}{\cfrac{\Sigma_1}{\Sigma_2}} = \frac{1}{x} + \cfrac{1}{\cfrac{2\cdot 1+1}{x} + \cfrac{1}{\cfrac{\Sigma_2}{\Sigma_3}}} = \;....\; = \frac{1}{x} + \cfrac{1}{\cfrac{3}{x} + \cfrac{1}{\cfrac{5}{x} + \cfrac{1}{\quad}}}$$

So as we see, the over complicated sums eventually disappeared from the final formula and we got a simple way to calculate $\frac{\Sigma_0}{\Sigma_1}$.

Of course an exact proof requires to show the existence of the limits. Our previous theorem does this for the right side because $\frac{3}{x}+\frac{5}{x}+\ldots = \frac{1}{x}(3+5+\ldots) = \infty$

2.)

The basic proof can be made exactly like in 1.) but the existence of limit for subtractive c.r.s. as we remember is much more complicated. Of course we have the simple $x_1, x_2, \ldots \geq 2$ condition, in our case: $\frac{3}{x}, \frac{5}{x}, \ldots \geq 2$ which simply means $x \leq \frac{3}{2}$. We might think to overcome this strong restriction by the following argument: For any x the $\frac{3}{x}, \frac{5}{x}, \ldots$ numbers eventually must become ≥ 2 so starting the c.r.s. from there, we get a limit and then the earlier terms can be used to calculate the final limit. This however would fail if an earlier first member, say $\frac{5}{x}$ would accidentally have the same absolute value as the negative limit after this, thus making the denominator 0 . Whether this is possible, I don't know.

R The real importance of Lambert's formulas can only be appreciated if we first turn back to a basic question about c.r.s.-s. For a given number, there are different c.r.s. that have this same number as limit. Indeed we can change x_1 and x_2 so that they compensate each other. If we use only whole numbers though, then the sequences are unique for each limit value.

T The limits of two additive or subtractive c.r.s. made from not all identical $x_1, x_2, \ldots$ and $z_1, z_2, \ldots$ sequences of positive whole numbers are different.

P Suppose that the limits were the same. In our simplified form:

$$\frac{1}{x_1 \pm \frac{1}{x_2 \pm}} = \frac{1}{z_1 \pm \frac{1}{z_2 \pm}} \quad \text{then: } x_1 \pm \frac{1}{x_2 \pm} = z_1 \pm \frac{1}{z_2 \pm}$$

This happens not as it seems by taking the reciprocal of both sides because the equation abbreviates limits. Still, reciprocal sequences have reciprocal limits so we get the same result.

With the same argument: $$x_1 - z_1 = \pm\frac{1}{z_2 \pm} \mp \frac{1}{x_2 \pm}$$

And so, for absolute values: $$|x_1 - z_1| = \left| \pm\frac{1}{z_2 \pm} \mp \frac{1}{x_2 \pm} \right|$$

But this is impossible if $x_1 \neq z_1$ because then their difference that is the left side is at least 1, while the right side is less than 1 . Similarly step by step all $x_n = z_n$ follows.

R

The uniqueness of additive or subtractive c.r.s.-s from wholes, suggests that these unique sequences of whole numbers should be somehow derived from the limit value. Indeed there is such derivation procedure, in fact this was historically before considering continued reciprocals made from real numbers. We preferred this modernized reverse order because otherwise the questions of limits would have been much more confusing. The name of Euclid in the following definition tells that it was him who discovered the heuristic way to find the whole values to get a limit by additive c.r.s. Such limit would be always a real number between 0 and 1 . To get a more general way for all real numbers, all we have to do is add an initial x_0 whole to the $\cfrac{1}{x_1 + \cfrac{1}{x_2 +}}$ c.r.s.

As it turns out this generalization not only fits perfectly into the procedure but suggests the "whole" idea, namely separating the whole part and the remainder, then repeat this for the reciprocal of the remainder and so on:

D

Let x be a positive real number:

1.) Remainder Forms

The basic remainder form of x is $x = x_0 + r_0$ with the x_0 whole part and r_0 decimal remainder. If $r_0 \neq 0$ it can be measured into 1 some x_1 many times with r_1 remainder, so: $\frac{1}{r_0} = x_1 + \frac{r_1}{r_0}$ This gives a new, so called first remainder form of x as:

$$x = x_0 + r_0 = x_0 + \cfrac{1}{\cfrac{1}{r_0}} = x_0 + \cfrac{1}{x_1 + \cfrac{r_1}{r_0}}$$

If $r_1 \neq 0$ it can be measured into r_0 some x_2 many times with r_2 remainder so: $\frac{r_0}{r_1} = x_2 + \frac{r_2}{r_1}$ This gives again a new second remainder form of x as:

$$x = x_0 + \cfrac{1}{x_1 + \cfrac{r_1}{r_0}} = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\cfrac{r_0}{r_1}}} = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{r_2}{r_1}}}$$

And so on, as long as an n-th remainder is not 0 we always have a next one.

If $r_n = 0$ then of course

$$x = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{1}{x_n + \cfrac{r_n}{r_{n-1}}}}} = x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{1}{x_n}}}$$

so this last n-th remainder form is actually a form giving x with only the whole parts.
If r_n is never 0 then we have infinite many remainder forms of x .

2.) Excess Forms

The basic excess form of x is $x = x_0 - e_0$ where x_0 is the next whole after x or x if it is a whole. e_0 can be called the excess of x . Similarly to the remainder forms we can define the excess forms containing minuses in places of the pluses.

3.) Euclidian Algorithm

The euclidian algorithm of x is the sequence: $x_0 \quad , \; x_0 + \frac{1}{x_1} \; , \; x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2}} \; , \; . \; . \; . \; .$

that is omitting the remainder parts from the remainder forms of x .
If x had only finite many remainder forms because $r_n = 0$ then the eucl. alg. is also finite and the last member is x itself as we showed above.

4.) Subtractive Euclidian Algorithm

The subtractive euclidian algorithm of x is similarly obtained from the excess forms.

T

1.) If the eucl. alg. of x is infinite then it approaches a value. Similarly for subtractive.

2.) This value is x itself.

3.) If $x = \frac{a}{b}$ fraction then its eucl. alg. is finite. Thus if the eucl. alg. of an x is infinite then x can't be a fraction, so it is a so called irrational number.
Same for sub. eucl. alg.

4.) If $x_1 > |y_1| \; , \; x_2 > |y_2| \; , \; . \; . \;$ are all wholes and the $\frac{y_1}{x_1} \; , \; \cfrac{y_1}{x_1 + \cfrac{y_2}{x_2}} \; , \; . \; . \;$ c.f.s.

approaches a limit then this limit is irrational.

5.) If $x_1 > z_1 \; , \; x_2 > z_1 z_2 \; , \; x_3 > z_2 z_3 \; , \; . \; . \;$ are positive wholes and the

$\pm \cfrac{1}{\cfrac{x_1}{z_1}} \; , \; \pm \cfrac{1}{\cfrac{x_1}{z_1} \pm \cfrac{1}{\cfrac{x_2}{z_2}}} \; , \; . \; . \; .$ c.r.s. approaches a limit then this limit is irrational.

6.) Lambert's sum ratios at x = fraction values are irrational.

P

1.) Enough to show that the c.r.s. without the initial x_0 term has limit.
For additive eucl. alg. : $x_1 + x_2 + \; . \; . = \infty$, for subtractive: $x_1 \; , \; x_2 \; , \; . \; . \; . \; \geq \; 2$

2.) $x = x_0 + r_0 > x_0$

$$x = x_0 + \cfrac{1}{x_1 + \cfrac{r_1}{r_0}} < x_0 + \frac{1}{x_1}$$

$$x = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{r_1}{r_0}}} < x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2}}$$

and so on, x is in between the odd and even members and since by 1.), these approach each other, their limit is x . Similarly for subtractive case.

3.) $\frac{1}{b}$ fits b times into 1 and a times into $x = \frac{a}{b}$

Thus $\frac{1}{b}$ will fit whole times into $r_0 = x - x_0 \cdot 1$ then again into $r_1 = 1 - x_1 \, r_0$ and then into $r_2 = r_0 - x_2 \, r_1$ and so on.

Since $r_0 > r_1 > r_2 > . \; . \; .$ and these are all multiples of $\frac{1}{b}$, they decrease with at least $\frac{1}{b}$. But infinite many times this would be impossible. Similarly for subtractive eucl. alg.

4.) If all the $y_1, y_2 \ldots$ are 1 or -1 that is for additive or subtractive c.r.s. the proof is very easy. Indeed, if the limit of our c.r.s. is x then the eucl. alg. of x gives back the $x_1, x_2 \ldots$ sequence because by 2.) the eucl. alg. of x has x as limit and as we proved earlier, $x_1, x_2 \ldots$ are unique. Then by 3.) x is irrational. For general c.f.s. the proof could be the same if we had an eucl. alg. for those. Well we do! Let $y_1, y_2 \ldots$ be given positive or negative wholes and x a positive real number. The signs of $y_1, y_2 \ldots$ will decide whether we use normal or subtractive eucl. alg. while their absolute values will be used to modify the crucial step of the eucl .alg. itself. This step was to change the $\frac{r_n}{r_{n-1}}$ fraction into $\frac{1}{\frac{r_{n-1}}{r_n}}$.

The purpose of this was to get a bigger than 1 fraction in the denominator so that a new whole part and remainder could be obtained. We can use any number a instead of 1 to achieve this because: $\frac{r_n}{r_{n-1}} = \frac{a}{\frac{a\ r_{n-1}}{r_n}}$. This means that r_n should be measured not into r_{n-1} as before but instead into $a\ r_{n-1}$. So $a \geq 1$ must be. If a is a whole then the same logic we used in 3.) shows that any common unit of r_{n-1} and r_n fits into r_{n+1} too, and so the generalized eucl. alg. can only be infinite for irrational x starting value. The number a, we used here can be the absolute values of $y_1, y_2 \ldots$ but we have to remember that the signs that decide whether remainder or excess should be calculated apply to the last two rem. or exc. while the absolute value applies when the new rem. or exc. is measured.
For example the sequence $y_1, y_2 \ldots = -2, 4, \ldots$ must be applied as follows:
The first minus sign tells us that the unit 1 must be measured over the initial x to get an e_0 excess and then the value 2 tells that this e_0 is measured into not the unit 1 but $2 \cdot 1$. Then the plus sign of 4 tells that e_0 must be measured under the unit and get an r_1 remainder which then will be compared to $4\,e_0$. And so on:

$$x = \frac{x}{1} = x_0 - \frac{e_0}{1} = x_0 - \frac{2}{\frac{2 \cdot 1}{e_0}} = x_0 - \frac{2}{x_1 + \frac{r_1}{e_0}} = x_0 - \frac{2}{x_1 + \frac{4}{\frac{4\,e_0}{r}}} = \ldots$$

This generalized eucl. alg. will always give $x_n \geq y_n$ if y_{n+1} is positive and $x_n \geq |y_n| + 1$ if y_{n+1} is negative. In reverse too, the crucial point of uniqueness can only be proved with these conditions that guarantee the step by step unique whole part and less than 1 remainder or excess. Our condition $x_n > |y_n|$ secured both the positive or negative cases.

5.) Follows from 4.) by:

$$\pm \frac{1}{\frac{x_1}{z_1}} = \frac{\pm z_1}{x_1} = \frac{y_1}{x_1}$$

$$\pm \frac{1}{\frac{x_1}{z_1} \pm \frac{1}{\frac{x_2}{z_2}}} = \frac{\pm z_1}{x_1 + \frac{\pm z_1 z_2}{x_2}} = \frac{y_1}{x_1 + \frac{y_2}{x_2}}$$

$$\pm \frac{1}{\frac{x_1}{z_1} \pm \frac{1}{\frac{x_2}{z_2} \pm \frac{1}{\frac{x_3}{z_3}}}} = \frac{\pm z_1}{x_1 + \frac{\pm z_1 z_2}{x_2 + \frac{\pm z_2 z_3}{x_3}}} = \frac{y_1}{x_1 + \frac{y_2}{x_2 + \frac{y_3}{x_3}}}$$

6.) From a point the fractions in the Lambert c.r.s. will satisfy the conditions of 5.) because x is a fixed fraction while the odd multipliers increase to infinity. Then of course if the limits are irrational from a point then they must be from the beginning already.

R

The whole goal of Lambert's sum ratio formulas was due to this last result, namely to show the irrationality of the sum ratios through the continued fraction forms. This is practical because many important functions and numbers can be expressed as sums or their ratios. In particular the number **e** directly, while π only through a very smart and tricky argument turns out to be irrational from Lambert's formulas. We might ask why all this fuss, after all if the euclidian algorithm of a number is infinite then it is irrational, so all we have to do is find this for **e** and π. Strangely the older π is a much harder case then the newer **e**. The greeks raised the question if a distance equal to the circumference of a circle, could be constructed from the radius with ruler and compass. This of course means constructing π from the unit. The actual calculation of π can be done by approaching the circle with polygons. Then it is not so surprising that the decimal form of
$\pi = 3.141592\,.\,.\,.$ lacks any rule. From the longer and longer calculated decimal form, we can also longer and longer calculate its euclidian algorithm which turns out like this:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \ddots}}}}}$$

So unfortunately no pattern emerges! By the way, the suddenly big value of 292 is a very lucky thing because it causes that $3 + \cfrac{1}{7 + \cfrac{1}{16}} = 3 + \frac{16}{113} = 3.14159292035\,.\,.$ is accurate up to six decimals.

Unlike at π, for the $e = \lim\left(1 + \frac{1}{n}\right)^n = 2.71828\,.\,.\,.$ we get a pattern in the eucl. alg.:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \ddots}}}}}}}}$$

This also suggests what above I said that Lambert's method can directly apply to **e**, while only indirectly to π. A strange fact is though, that already more than hundred years before Lambert, Lord Brouncker the first president of the royal society published a c.f.s. for π:

$$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \ddots}}}}}}$$

He didn't give any proof but we'll give it later following Euler. The more important missing part was any implication about the irrationality of π. Since this was part of an ancient problem of mathematics, it couldn't just slip his mind. Probably he struggled to make a connection from his form to the irrationality, but couldn't succeed. Indeed his continued fraction is not a generalized euclidian algorithm because the y numerators are increasing far beyond the fixed 2 x-values. The real amazing fact is that since Brouncker nobody else could come up with a gen. eucl. alg. for π. Probably it is impossible with patterned y-s and x-s but nobody proved this yet either. Strangely only two hundred years after Brouncker did it turn out that even simpler or more basic non euclidian algorithmic continued fractions exist for π too. My favorite is:

$$\pi = 2 + \cfrac{2}{1 + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\frac{1}{4} + \cfrac{1}{\frac{1}{5} + \ddots}}}}}$$

Though this is totally impractical because very slowly approaching, the fact that a c.r.s. with all the reciprocals gives π is amazing. By the way it is a nice example for the infinity of sums of reciprocals as the limit condition of the additive c.r.s.-s.
The stubborn resistance of π against to be analyzed is a more general feature and best shown by the fact that the solution of the original greek problem, namely the proof of the non constructibility of π, only succeeded at the end of the 19-th century.

1. Newton-Leibniz rule

R

Differentiation and integration were discovered by Newton and Leibniz about the same time. The fundamental principle that connects the two concepts is called today the Newton-Leibniz rule, thus combining the names of the two persons who in real life spent their years in a bitter fight . It was the stupidity of some “admirers” of Newton in the royal academy that drove the greatest physicist into the lowest level of personal attacks against Leibniz.
Newton needed this mathematics for his physics while Leibniz discovered it from purely mathematical and philosophical motivations. The principle of the Newton-Leibniz rule is so simple and basic that we all use it daily without even noticing. So we could even call it an a-priori or instinctive ability and I wouldn’t be surprised if it turned out that even animals use it. I’ll explain this soon but first we have to emphasize that the third name of differentiation and integration, the so-called calculus is throwing an important light on why only so late in history became conscious such a basic principle. Calculus means exactly what it says, that is calculating. So beside the big connection of the Newton-Leibniz rule we are able to actually calculate differentials and integrals for a sequence of concrete functions. Then the connecting rule becomes verified through infinite many concrete examples thus giving practical reality to the pure principle. In the next section I’ll emphasize this through one particular sequence of examples, namely the power functions $y = x^n$. Formalism only accepts such practical results and stays away from describing the pure principles to us. How wrong formalism is, therefore will be shown by how much someone appreciates the following explanations that try to throw light on the intuitive meaning of Newton-Leibniz rule. I start with the simplest example:
If we ask a shopkeeper how was his day, he might react in two ways. He can simply recollect from his memory how business went or he might go to the cashier, count the money and give an exact answer. The interesting thing is that when he estimates the business then he is not using his memory in the second manner, in fact he might not even have counted the cashier all day. Our memory remembers the “flows” of events like “there was a lot of tourists buying around ten o’clock then two hours flat time then again somebody”, and so on. Our brain can intuitively translate the memories of past flows into our present situations. We can of course check our present situations by direct observations too and so the other process could be called a biological approximation which is continuous and automatic in consciousness. If we regard only numerical changes then the previous money income was a good example though we should allow the outcome that is spending too. Or in general, objects have values too so at every moment a shop has its value that can be measured by stocktaking. The flow of business then is the in and out going value. In fact a double flow happens because the shop is in balance, everything must be paid by money or credit notes. This doubling is just a social process and real life always has unbalanced changes like presents or losses. The main question is how can we approximate values, that is results of a potential stocktaking. As I said, we remember the flows of traffic but this expression still hides two vital elements namely that this flow is actually the rate of density of the traffic and that we have to remember the times, in particular the time intervals how long those flows kept on. Indeed if we remember that for a very short time very high inflow happened like a couple buying a lot, then this can worth more then a lot of kids buying chewing gum later. So our brain multiplies these two factors, that is how long and how big flows happened and this gives the actual changes. Finally of course the obvious process is to add up all these elemental changes and come up with a total. Now it’s quite easy to formulate these:
Let x denote the time and $y(x)$ a quantity that changes with time.
We don’t store $y(x)$ in our memory, instead its change <u>rate</u> $y'(x)$. If we want to know the <u>actual</u> change in y from time a to time b that is $y(b) - y(a)$ then this is approximated as follows:
We divide the time between a and b as: $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$.
Then every small difference is multiplied with the change rate at that time, and all these added:

$$(x_2 - x_1)\ y'_1 + (x_3 - x_2)\ y'_2 + \ldots + (x_{n+1} - x_n)\ y'_n = \sum_1^n (x_{k+1} - x_k)\, y'_k$$

Unfortunately there is a big problem here! We said: "the change rate at that time" but "that time" is not specified. Even though x_k and x_{k+1} are close, there are infinite many points between them so if the y' rate of flow is changing continually then y'_k is actually a changing $y'(x)$ function between x_k and x_{k+1}. To be really general, we should pick arbitrary t time points between the x ones as: $a = x_1 \le t_1 < x_2 \le t_2 < \ldots < x_n \le t_n < x_{n+1} = b$

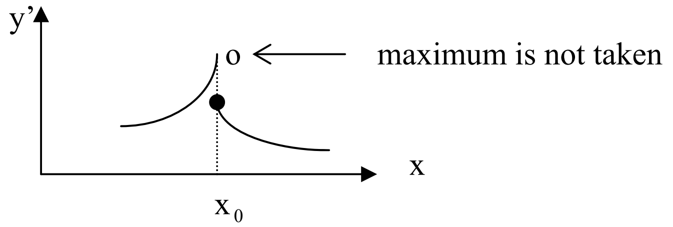
Then we can really say that:

$$y(b) - y(a) \approx (x_2 - x_1)\ y'(t_1) + \ldots + (x_{n+1} - x_n)\ y'(t_n) = \sum_1^n (x_{k+1} - x_k)\, y'(t_k)$$

Amazingly, the original problem of why this approximation works and how y' comes from y, will easily follow from solving this seemingly minor difficulty that is choosing the t time points in the small intervals. So first we show that:

If we use finer and finer divisions between the a , b times, then the choices of t-s are immaterial.

Of course we have to be a bit more precise about "finer and finer". To increase the number of $x_1, x_2, \ldots x_n$ points is not enough, we have to make sure that they get closer to each other, that is their consecutive differences tend to 0. Other assumptions about the $y'(x)$ change rate are also needed namely, first that it has a maximal and minimal value on any of the small $[x_k, x_{k+1}]$ time intervals. The following figure shows that this doesn't follow automatically:



We might say: Of course y' was not continuous, it was disconnected. Indeed we could go into the problems of continuity or connection and then prove the existence of maximum and minimum. But we ignore these side fields now and just assume the maximum and minimum. So lets denote these on the $[x_k, x_{k+1}]$ interval as $\overline{y'_k}$, $\underline{y'_k}$ and also $(x_{k+1} - x_k)$ as dx_k. Then:

$$dx_1\, \underline{y'_1} + \ldots + dx_n\, \underline{y'_n} \le dx_1\, y'(t_1) + \ldots + dx_n\, y'(t_n) \le dx_1\, \overline{y'_1} + \ldots + dx_n\, \overline{y'_n}$$

To see that the $t_1, t_2, \ldots t_n$ chosen time instances are immaterial, it is enough to show that the left and right minimal and maximal sum values approach each other, that is: D =

$$(dx_1\, \overline{y'_1} + \ldots + dx_n\, \overline{y'_n}) - (dx_1\, \underline{y'_1} + \ldots + dx_n\, \underline{y'_n}) = dx_1\, (\overline{y'_1} - \underline{y'_1}) + \ldots + dx_n\, (\overline{y'_n} - \underline{y'_n}) \to 0$$

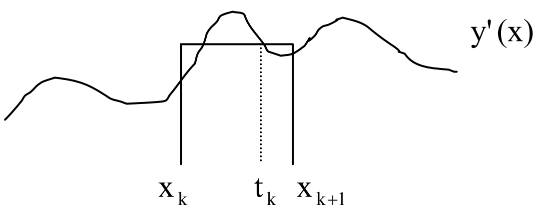
To show this, lets pick the biggest of the maximum, minimum difference say $(\overline{y'_k} - \underline{y'_k})$ and then replacing this into the others we clearly increase the total D difference so:

$$D \le (\overline{y'_k} - \underline{y'_k})\ (dx_1 + dx_2 + \ldots + dx_n) = (\overline{y'_k} - \underline{y'_k})\ (b - a)$$

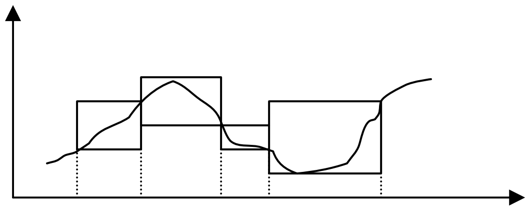
Since the $dx_k \to 0$ thus the possible y' values on dx_k must be getting closer to each other including the maximal and minimal, that is $(\overline{y'_k} - \underline{y'_k}) \to 0$ and thus $D \to 0$ too.

Here we again made an assumption that is not obvious and related to continuity but again instead of examining it lets go on to something more important. I want to draw attention to the fact that our coordinate system representation gives a whole new meaning to this approximated sum as area. The word integral refers to the integrated that is summed up little approximation.

The name does not refer to what was integrated, that is to the y' continuous flow rate. In other words y' can be regarded in itself as a given function to start with. Then the original idea of integrated approximation is lost and only the special relation of x to y' will define the meaning. If we draw y' as any given function in a coordinate system then the y' values are perpendicular to the x differences and so their $(x_{k+1} - x_k)\, y'(t_k)$ products will give the areas of rectangles with $[x_k, x_{k+1}]$ base and $y'(t_k)$ height:

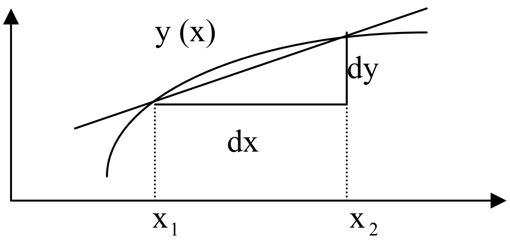


The maximal , minimal y'-s are heights, that define the highest and lowest rectangles.
Their difference D, can be seen as the sum of rectangles containing the curve of $y'(x)$:

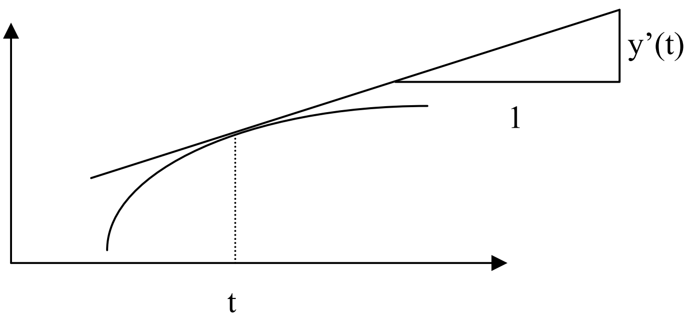


This gives a visual representation of the proved $D \to 0$ because with more and more refined intervals, the containing rectangles get thinner and thinner and “finally” become the curve itself which has no area. The main thing is of course that the total integral becomes the area under the function. This secondary meaning, that came out of coordinate system became over emphasized and regarded as the fundamental one. When students get to higher concepts of integration they have to break away from this and rediscover the real meaning that we started with.
Returning to this, we have to find the meaning of y' from y and then choose $t_1, t_2, \ldots, t_n$ so that the approximation for $y(b) - y(a)$ becomes apparent. Just to clear up the names:
Integration is integrated (summed up) approximation with a y' change rate while differentiation is the derivation of y' from y. No wonder that the word “derivative” is also used for y' but the particular values of it are called the differential ratios. This name is very revealing but lets start with an everyday paradox. After an accident, the police may say that the car was traveling with 120 kilometer per hour (in short 120 km/h) speed when it hit the curb and turned over. We heard these so many times that we never stop and wonder how the concept of kilometer can be even mentioned when the whole accident took place in seconds. If someone thinks about it, then of course he can say that the 120 km/h meant that the car would have traveled 120 km in a full hour if the accident hadn’t stopped it and it had kept on going. We can even mention that the 120 km/h is just a way of saying how fast the car was and we can express it in meters and seconds and any other units. As usual, every calmly over simplified explanation just brings in even more hidden problems. Indeed, we used two lines of logic, one about how the car would have traveled if the accident didn’t happen and one about other possible measurements of the speed. Apart from going into the philosophical problems with expression like “would have happened”, even physically it’s hard to tell how far we have to remove the circumstances to regard the car going “by itself”. We might think that the road is clearly needed “to go” but that is changing the travel itself. All this leads to Newton’s physics, the discovery of inertia, forces and so on. As it turns out the road is not necessary!!! If we remove everything outside the car, even the earth and the sun then it will keep traveling with its 120 km/h in empty space.
The other approach, that is sticking more to reality is strangely the mathematical one:

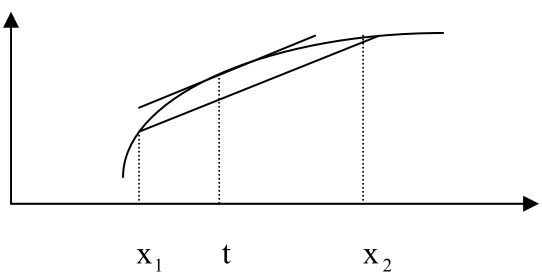
Here we can say that getting closer and closer to the moment of accident from the past, we can calculate the average speeds simply as distances divided by time intervals and the closer we get to the point the more and more accurate value we get for the final 120 km/h. So again we ended up with an approximation but now we don't approximate intuitively like the integration was, instead we introduce it as a purely theoretical tool to derive the intuitively already given speed or any other rate of change. When we see moving objects or any changes like water flowing, wind blowing, we perceive the rate of these changes directly and then we can approximate from these the long time effects by integration. If it turns out that I am wrong and actually our perception does a differential ratio approximation too, then it's an even better reason to study the followings:
The differential ratio or change rate limit also becomes easy to see in a coordinate system. Again x is the time coordinate and y is a measured quantity.
The change ratio is actually the slope of the connecting line between two points:



$\frac{dy}{dx} = y'(x_1, x_2)$ is the approximative change ratio (difference ratio) and when the $[x_1, x_2]$ interval shrinks to a t point then it becomes the slope of the tangent line at t (differential ratio):



The triangle that we drew on the tangent line is just an easy way to see the slope because if we choose the x side as 1, then the y side is the slope itself.
Lets remember that we already used two assumptions about y'. The first was the existence of its maximums and minimums in all the $[x_k, x_{k+1}]$ intervals, the second was that the maximums and minimums approach each other as the intervals shrink. Now we make a third assumption that can be seen very well in a coordinate system. If a function's graph connects two points of the coordinate system, one at x_1 and an other at x_2 time, then there is always a t time between them at which the tangent has the same slope as the connecting line between x_1 and x_2:



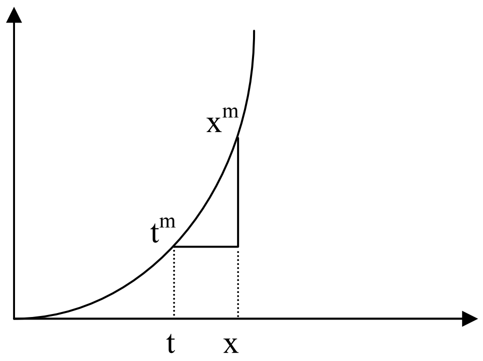
Geometrically this means parallelity, while algebraically that $y'(t) = \frac{y_2 - y_1}{x_2 - x_1}$.

Applying this to all the little intervals in the integration we get what we wanted:
$(x_2 - x_1)y'(t_1) + (x_3 - x_2)y'(t_2) + \,.\,.+ (x_{n+1} - x_n)y'(t_n) = \not{y}_2 - y_1 + \not{y}_3 - \not{y}_2 + \,.\,. + y_n - \not{y}_{n-1} =$
$y_n - y_1 = y(b) - y(a)$
We have to realize that nothing mysterious happened. The integration uses the y' change rates as multipliers to get the changes on the small $[x_k , x_{k+1}]$ intervals and the y' change rates are derived as limits of difference ratios. So it's quite expectable that we get the correct approximation for the total change of y. On the other hand it is not trivial that we got this result because the limits of difference ratios that is differential ratios are calculated for every point and then the integration uses any of these. So what the Newton-Leibniz rule really says is that the two kinds of limit taking, the differential and the integral stay in synchrony and the expectable connection remains. In practice the meaning is more striking because the two kinds of limits can be calculated independently and then it is an always reappearing "coincidence" that the integral of the derivative gives back the original function: $\int_a^b y'\,dx = y(b) - y(a)$
The $\int$ sign is a "smoothened version" of the $\sum$ sum sign because of its limit value and the dx is merely a symbol, in one sense to remind us that $(x_{k+1} - x_k) = dx_k$ were multiplied with the $y'(t_k)$ but more importantly to show the variable x and the end of the integral for cases when y' is a more complicated, longer expression.
In the next section, we'll show an example of the mentioned "coincidence".

2. Powers

T The tangent slope of the $y = x^m$ power function at an x value is $y' = m \cdot x^{m-1}$

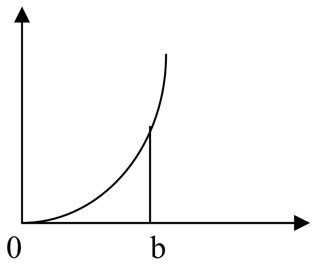
P Instead of the notation that we used in the previous section, where $[x_1, x_2]$ shrank to a t value, here we simplify the situation and x will be the limit point approached by a single t.
In other words the $[t, x]$ interval shrinks to x.



$$y'(x) = \lim_{t\to x} \frac{x^m - t^m}{x - t} = \lim_{t\to x} (x^{m-1} + x^{m-2}\, t + x^{m-3} t^2 + \ldots + x\, t^{m-2} + t^{m-1}) = m\, x^{m-1}$$

R As we see, the typical trick to find derivatives, is to replace the theoretical difference ratio with an other one. The obvious practical reason for this is that t cannot be simply replaced by x in the theoretical ratio because then the denominator would become 0. Thus in fact the replaced tricky expression cannot even be identical with the theoretical one. Indeed the replaced function is usually an extended version of the difference ratio. Thus the old and usually neglected nuances of losing and gaining solutions to equations in algebra, here become the crucial elements.

T The area under the $y = x^m$ power function between 0 and x is $\int_0^b x^m dx = \frac{b^{m+1}}{m+1}$



P Lets use on the [0,b] interval equal dividing points:

$x_1 = 0, x_2 = \frac{b}{n}, x_3 = 2\cdot\frac{b}{n}, \ldots\ x_{n+1} = n\cdot\frac{b}{n} = b$ and use $t_1 = x_2, t_2 = x_3, \ldots$
right end value points. Then the sum of the rectangle areas is:

$$\frac{b}{n}\left(\frac{b}{n}\right)^m + \frac{b}{n}\left(2\cdot\frac{b}{n}\right)^m + \ldots + \frac{b}{n}\left(n\cdot\frac{b}{n}\right)^m = \left(\frac{b}{n}\right)^{m+1}\left(1 + 2^m + 3^m + \ldots + n^m\right) =$$

$$b^{m+1}\frac{1 + 2^m + \ldots + n^m}{n^{m+1}}$$

All we have to show is that this fraction approaches $\frac{1}{m+1}$ as n increases and the trick to do this is to put an m-th power of a k number between two values, so observe that:

$$\frac{k^{m+1}-(k-1)^{m+1}}{m+1} \quad < \quad k^m \quad < \quad \frac{(k+1)^{m+1}-k^{m+1}}{m+1} \qquad \text{because:}$$

$$k^{m+1}-(k-1)^{m+1} = \underbrace{[\,k-(k-1)\,]}_{1}\left\{k^m + \underset{\wedge \atop k^m}{k^{m-1}(k-1)} + \ldots + \underset{\wedge \atop k^m}{k\,(k-1)^{m-1}} + \underset{\wedge \atop k^m}{(k-1)^m}\right\} < (m+1)k^m$$

and

$$(k+1)^{m+1}-k^{m+1} = \underbrace{[\,(k+1)-k\,]}_{1}\left\{\underset{\vee \atop k^m}{(k+1)^m} + \underset{\vee \atop k^m}{(k+1)^{m-1}k} + \ldots + \underset{\vee \atop k^m}{(k+1)k^{m-1}} + k^m\right\} > (m+1)k^m$$

Using our inequality for $k = 1, 2, 3, \ldots n$ we get:

$$\frac{1^{m+1}-0^{m+1}}{m+1} \quad < \quad 1^m \quad < \quad \frac{2^{m+1}-1^{m+1}}{m+1}$$

$$\frac{2^{m+1}-1^{m+1}}{m+1} \quad < \quad 2^m \quad < \quad \frac{3^{m+1}-2^{m+1}}{m+1}$$

$$\vdots$$

$$\frac{n^{m+1}-(n-1)^{m+1}}{m+1} \quad < \quad n^m \quad < \quad \frac{(n+1)^{m+1}-n^{m+1}}{m+1} \qquad \text{adding these together:}$$

$$\frac{n^{m+1}}{m+1} \quad < \quad 1^m + 2^m + \ldots + n^m \quad < \quad \frac{(n+1)^{m+1}-1}{m+1} \qquad \text{dividing by } n^{m+1}:$$

$$\frac{1}{m+1} \quad < \quad \frac{1^m + 2^m + \ldots + n^m}{n^{m+1}} \quad < \quad \frac{\left(1+\frac{1}{n}\right)^{m+1} - \frac{1}{n^{m+1}}}{m+1} \quad \rightarrow \quad \frac{1}{m+1}$$

3. Reciprocal

R

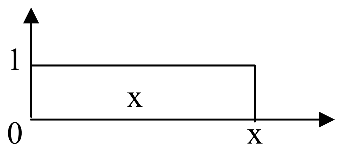
The results $(x^n)' = n\,x^{n-1}$ and $\int_0^b x^n dx = \dfrac{b^{n+1}}{n}$ are examples of Newton-Leibniz rule.

This can be expressed even more obviously if in the $\int_0^b x^n dx$ integral b is regarded as the x variable. Then of course we should use a new say t variable for x but this new $\int_0^x t^n dt$ form can be abbreviated as $\int x^n$ and then $(x^n)' = n\,x^{n-1}$ and $\int x^n = \dfrac{x^{n+1}}{n}$ look really "opposites". The derivative of x^n is: "reducing the exponent and multiplying with the old", while the integral is "increasing the exponent and dividing by this new". The true meaning of "opposites" of course should be $\int y' = y$ and indeed $\int (x^n)' = \int n\,x^{n-1} = n\dfrac{x^{(n-1)+1}}{(n-1)+1} = x^n$. Here we had to use that $\int n\,.\,. = n\int .\,.$ but this is obvious if we think of the rectangular sums. This even more formal inversion of derivatives and integrals might suggest that there is a perfect back and forth correspondence. Then the totally different ways we can obtain the actual rules for derivatives and integrals seems even more surprising. This is not true, there is no perfect correspondence! There are derivative rules which have no "opposite" integral reversals.

As we said, derivative tricks are much easier than integrals. This also goes for generalizing them. Best example is the power functions for which the simplest extension could be the negative whole exponents, that is $y = x^{-n} = \dfrac{1}{x^n}$. To prove that the rule remains is easy:

$$\left(\frac{1}{x^n}\right)' = \lim_{t\to x}\frac{\frac{1}{x^n}-\frac{1}{t^n}}{x-t} = \lim_{t\to x} -\frac{\frac{x^n-t^n}{x-t}}{x^n\,t^n} = -\frac{\lim\limits_{t\to x}\frac{x^n-t^n}{x-t}}{x^{2n}} = -\frac{n\,x^{n-1}}{x^{2n}} = -\,n\,x^{-n-1}$$

This result shows even more how "perfect" the definition of negative exponents was as reciprocals. In fact even the 0 exponent as 1 fits perfectly into the derivation because: $1' = (x^0)' = 0\,x^{0-1} = 0$. Indeed the $y = 1$ constant function has 0 slope everywhere. The integration can also be used as reversal for 0 exponent: $\int 1 = \int x^0 = \dfrac{x^{0+1}}{0+1} = x$. Indeed the area under the constant 1 function up to an x is x:



So where is the promised discrepancy? Lets try to integrate $\dfrac{1}{x} = x^{-1}$: $\int x^{-1} = \dfrac{x^{-1+1}}{-1+1} = \dfrac{x^0}{0} = ?$

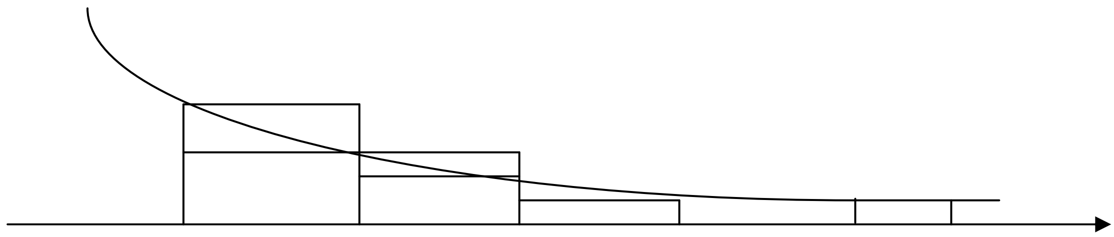
Division with 0 is not defined, so we didn't get an answer. And indeed:

$$y \;=\; \ldots\, x^3\;,\quad x^2\;,\quad x^1 = x\;,\quad x^0 = 1\;,\quad x^{-1} = \frac{1}{x}\;,\quad x^{-2} = \frac{1}{x^2}\;,\;\ldots$$

$$y' \;=\; \ldots\, 3x^2\;,\quad 2x\;,\quad 1\;,\quad 0\;,\quad -\frac{1}{x^2}\;,\quad -\frac{2}{x^3}\;,\;\ldots$$

So as we see the $\dfrac{1}{x}$ function is missing in the second line of derivatives and thus $\int \dfrac{1}{x}$ is not in the first line. In short $\int \dfrac{1}{x}$ leads out of the power functions. Thus we have to find it by the rectangular sums, just as for the power functions.

D Inner and outer rectangular sums for $y = \frac{1}{t}$, with equal d bases.



$$\sum \text{in} = d\left(\frac{1}{a+d}+\frac{1}{a+2d}+\ldots+\frac{1}{a+\left[\frac{b-a}{d}\right]d}\right) \quad , \quad \sum \text{out} = \frac{d}{a}+\sum \text{in}$$

T Proportional invariance. (By beginning proportional equal based rectangular sums)

1.) If $d = \frac{a}{m}$ then: $\sum \text{in} = \frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{\left[\frac{m\,b}{a}\right]}$, $\sum \text{out} = \frac{1}{m}+\sum \text{in}$

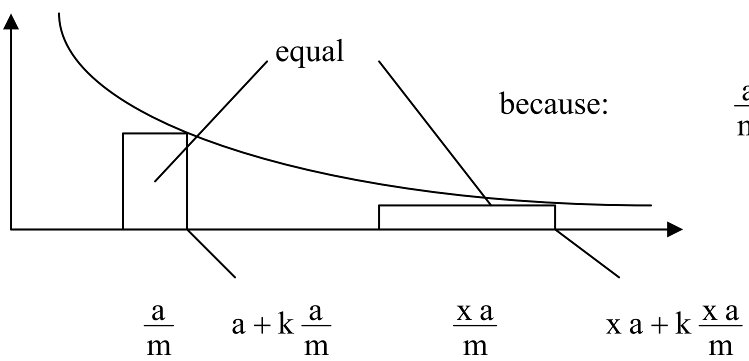
2.) $\int_a^b \frac{1}{t}\,dt = \int_{xa}^{xb} \frac{1}{t}\,dt$

P 1.) $\sum \text{in} =$

$$\frac{a}{m}\left(\frac{1}{a+\frac{a}{m}}+\frac{1}{a+2\frac{a}{m}}+\ldots+\frac{1}{a+\left[\frac{b-a}{\frac{a}{m}}\right]\frac{a}{m}}\right) = \frac{a}{m}\left(\frac{m}{a}\,\frac{1}{m+1}+\frac{m}{a}\,\frac{1}{m+2}+\ldots+\frac{m}{a}\,\frac{1}{m+\left[\frac{m(b-a)}{a}\right]}\right)$$

and $m+\left[\frac{m(b-a)}{a}\right] = m+\left[\frac{m\,b}{a}-m\right] = \left[\frac{m\,b}{a}\right]$ $\quad \sum \text{out} = \frac{\frac{a}{m}}{a}+\sum \text{in} = \frac{1}{m}+\sum \text{in}$

2.) $\sum$ in and $\sum$ out only depend on $\frac{b}{a}$. But we can see it directly rectangle by rectangle too:



because: $\frac{a}{m}\,\frac{1}{a+k\frac{a}{m}} = \frac{x\,a}{m}\,\frac{1}{x\,a+k\frac{x\,a}{m}}$

R $\int_a^{xb} \frac{1}{t} dt - \int_a^{b} \frac{1}{t} dt = \int_b^{xb} \frac{1}{t} dt = \int_1^{x} \frac{1}{t} dt$ = fixed amount that only depends on x.

So the areas measured from 1 to x have special meanings.
They show the fixed increase that happens if we change an [a , b] interval to [a , xb].
The fact that multiplying the upper end causes only an addition, reminds us of the exponents, where the opposite is true, an addition causes multiplication: $t^{r+s} = t^r t^s$.

Thus we expect $\int_1^{x} \frac{1}{t} dt$ to be related to an exponential function.

T

1.) $\frac{1}{m+1} + \ldots + \frac{1}{[m x]} < \int_1^{x} \frac{1}{t} dt < \frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{[m x]}$

2.) $\frac{1}{m+1} + \ldots + \frac{1}{m n} < \int_1^{n} \frac{1}{t} dt < \frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{m n}$

3.) $\frac{1}{2} + \ldots + \frac{1}{n} < \int_1^{n} \frac{1}{t} dt < 1 + \frac{1}{2} + \ldots + \frac{1}{n}$

4.) $\lim \int_1^{n} \frac{1}{t} dt = \int_1^{\infty} \frac{1}{t} dt = \infty$ but: $\lim \frac{\int_1^{n} \frac{1}{t} dt}{n} = 0$

P

1.) , 2.) , 3.) are trivial from 1.) of prev. theorem with a = 1 and b = x or n and m = 1

4.) The first is trivial by 3.) and for the second: $\frac{\int_1^{n} \frac{1}{t} dt}{n} < \frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}{n} \to 0$

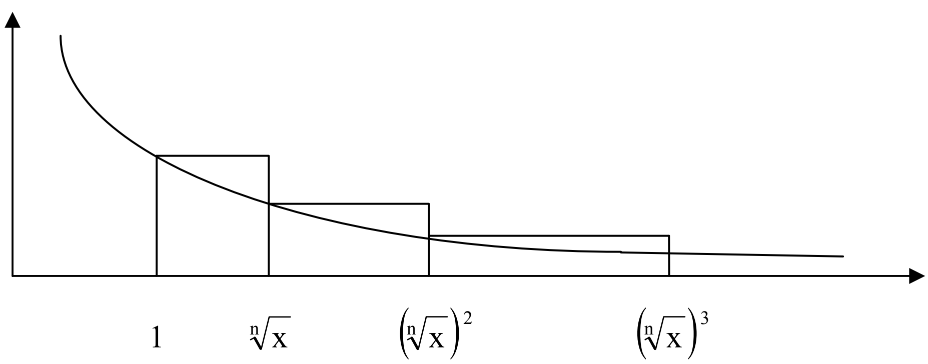
T

Explicit surd-limit form of area. (By proportionally increasing based rectangular sum)

$$\int_1^{x} \frac{1}{t} dt = \lim n\,(\sqrt[n]{x} - 1)$$

P

Lets place in the [1 , x] interval, dividing points that increase in the same rate.
Thus if the n-th is x itself, then the first must be $\sqrt[n]{x}$, the second $\left(\sqrt[n]{x}\right)^2$ and so on:

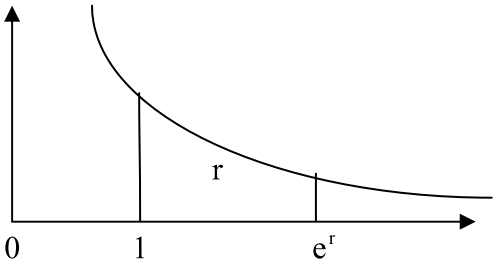


$$\Sigma\,\text{out} = \sqrt[n]{x} - 1 + \underbrace{\left[\left(\sqrt[n]{x}\right)^2 - \sqrt[n]{x}\right] \frac{1}{\sqrt[n]{x}}}_{\sqrt[n]{x} - 1} + \underbrace{\left[\left(\sqrt[n]{x}\right)^3 - \left(\sqrt[n]{x}\right)^2\right] \frac{1}{\left(\sqrt[n]{x}\right)^2}}_{\sqrt[n]{x} - 1} + \ldots = n\,(\sqrt[n]{x} - 1)$$

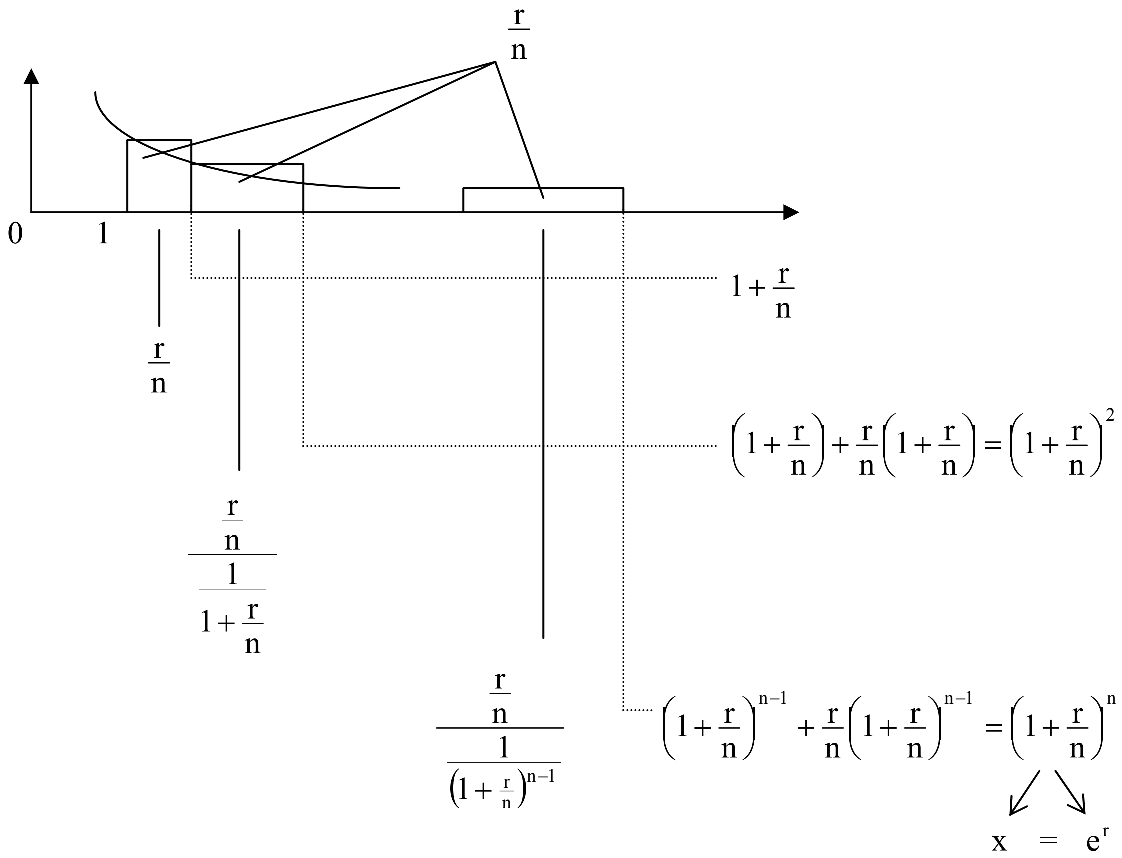
So as we see, we obtained equal rectangular areas in this sum.

T Explicit exponential form of the x end from the r area. (By $\frac{r}{n}$ rectangular sums)

If $\int_1^x \frac{1}{t} dt = r$ then $x = \lim \left(1 + \frac{r}{n}\right)^n = e^r$



P Again we use equal outer rectangular areas, but now they are n-th of the actual r area:

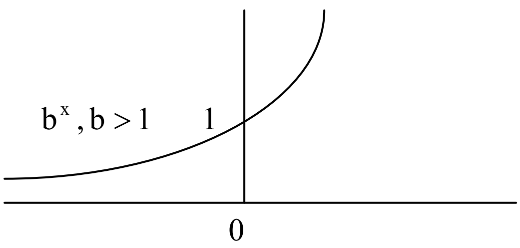


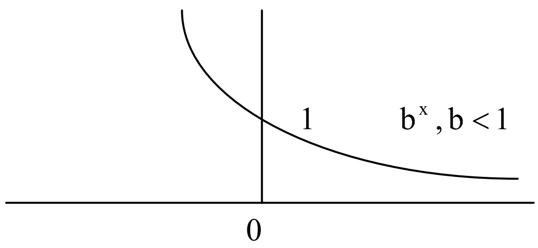
4. Logarithm

R

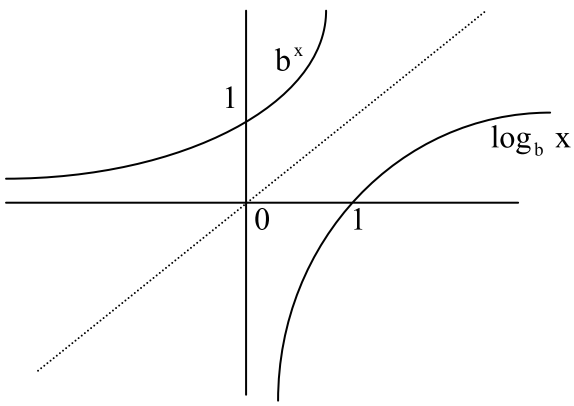
Our final result of the previous section still didn't give exactly the area under $y = \frac{1}{x}$.
In fact it gave the opposite, that is the upper bound as function of the r area measured from 1. This was $x = e^r$ and from this we can easily get the answer to our original question. Indeed if we regard the inverse of $x = e^r$ then this $r = \ln x$ gives the area that is $\int_1^x \frac{1}{t} dt = r = \ln x$.

This, so called natural logarithm function was already mentioned at the Compound Limit in the Infinite Sums of Reciprocals.
ln x is the exponent that **e** must be raised, to get x. In short $x = e^{\ln x}$.
The same obvious defining rule for other b based logarithm is $x = b^{\log_b x}$.
The uneasiness we feel at using logarithm is well based because there was a hidden jump in our logic. We only defined rational exponents namely the reciprocals with surds and the negative exponents with reciprocals. To take the inverse of exponentiation we must have unique powers for all exponents. Both the uniqueness and generalization to all exponents follows from the fact that b^r is monotone for the rational r exponents. Thus the values of b^x for non rational x "fill in" by limits. For $b > 1$ and $b < 1$ we get two oppositely monotone functions.





The inverse of this function, $\log_b x$ can also be seen as the mirror image to the $y = x$ 45^o line.



Similarly for $b < 1$

The proportional additiveness is a general feature of all logarithms: $\log_b (x\,z) = \log_b x + \log_b z$

Indeed: $x\,z = b^{\log_b xz} = b^{\log_b s} b^{\log_b z} = b^{\log_b x + \log_b z}$ implies this because the exponentiation is monotone, thus unique, so we can infer equality of exponents from equality of powers with same base. This hidden logic leads to even more surprising results:

T

1.) Changing base:

$$\log_B x = \frac{\log_b x}{\log_b B}$$

2.) Constant fraction of logarithm as new base logarithm

$$\frac{\log_b x}{c} = \log_{b^c} x \quad \text{in particular} \quad \frac{\ln x}{c} = \log_{e^c} x$$

3.) Changed scale reciprocal area as changed base logarithm:

$$\int_1^x \frac{1}{ct}\,dt = \log_{e^c} x$$

P

1.) $x = B^{\log_B x} = \left(b^{\log_b B}\right)^{\log_B x} = b^{\log_b B \log_B x} = b^{\log_b x}$

so uniqueness of exponents gives it.

2.) 1.) with $B = b^c$ gives it.

3.) $\int_1^x \frac{1}{ct}\,dt = \frac{1}{c}\int_1^x \frac{1}{t}\,dt = \frac{\ln x}{c} = \log_{e^c} x$ by 2.)

R

The proportional additiveness of logarithm is making it the most important function for physics or science in general. Before the computers and calculators tables of logarithms were used in high schools to “help” calculations. Indeed instead of multiplying x and z, the students looked up the logarithms of them, added these and then found the “back logarithm”, that is the power. This exactly means what we showed above: $x\,z = b^{\log_b x + \log_b z}$. The slide rule is an even more striking application with the addition performed as measuring distances after each other. This “instrument” of course belongs to the past too. In those days the obsession with the procedures left the real importance of logarithm unmentioned. Today ignorance achieves the same. This is quite amazing because logarithm surrounds us both in nature and human creation. The shape of a piano is direct consequence of the exponential function. The keyboard seems as a linear order of tones but a fixed increase in pitch means a multiple of frequencies and thus a division in the length of strings. This raises the question why we “hear” this way and then we get the even more amazing discovery that not only the pitch but the loudness too, works this way. Twice as big sound must only cause an addition of sense. Indeed otherwise we couldn’t hear both whispers and thunders with the same organ. But this line of reasoning goes even further to the processing of any information and thus leads to the exponentiation and logarithm to be used in measuring the entropy or “unevenness” of any system. Finally, we mention that:

Our surd-limit explicit form for $\int_1^x \frac{1}{t}\,dt$ gives an explicit form for ln x too:

T

$$\ln x = \lim n\,(\sqrt[n]{x} - 1)$$

5. Self area and irrationality

R

The previous remark might give the impression that logarithm is at least as important if not more than exponentiation. The followings show that in a sense exponentiation still “rules”:

T

1.) If $f(x) = \int_a^x f(t)\,dt$ then $f(x) = 0$

For every a , c there is only one f (x) that $f(x) = c + \int_a^x f(t)\,dt$

2.) If $f(x) = 1 + \int_0^x f(t)\,dt$ then $f(x) = \lim\left(1+\frac{x}{n}\right)^n$

3.) $\lim\left(1+\frac{x}{n}\right)^n = e^x$

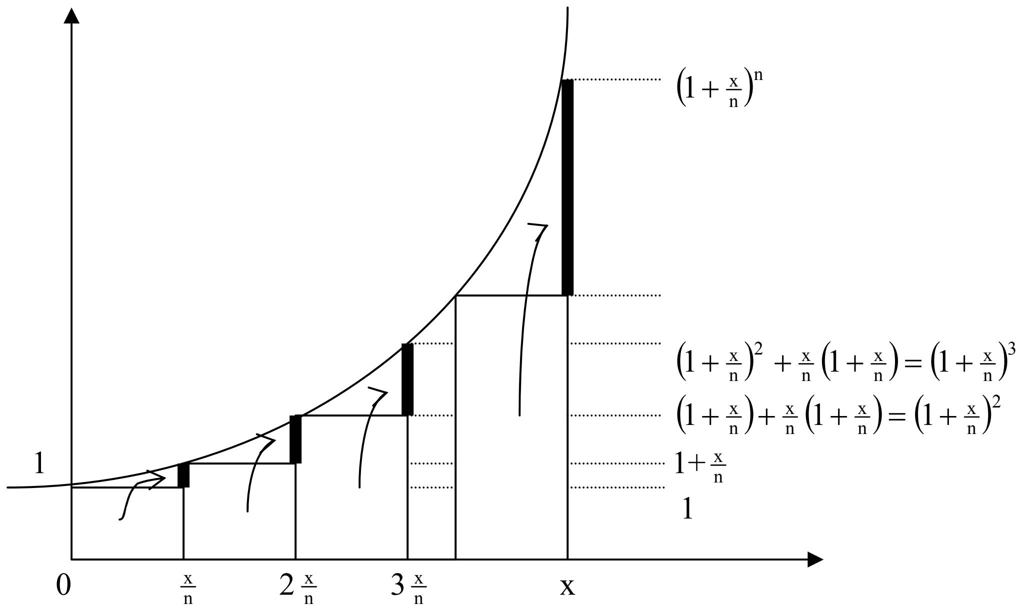
4.) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \ldots$

P

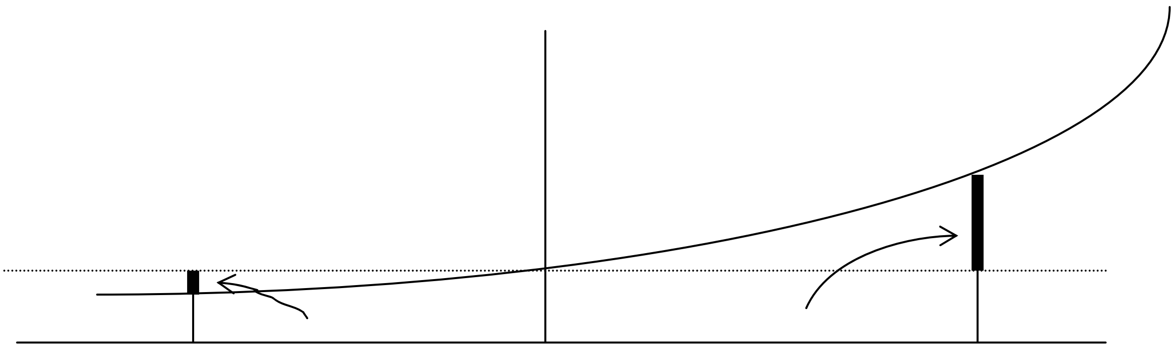
1.) $f(a) = \int_a^a f(t)\,dt = 0$

$f(a+\varepsilon) = \int_a^{a+\varepsilon} f(t)\,dt \le \varepsilon\, f(a+\varepsilon)$ can only be if $f(a+\varepsilon) = 0$ too.

2.) Let’s try to construct an f (x) function that starts with f (0) = 1 and increases as its area.



Now we can confirm similarly that this same function will decrease from f (0) = 1 towards the minus values with its area. So in the end the full f (x) function looks like:



This means that for negative x values $f(x) = 1 - \int_x^0 f(t)\,dt$ but if we agree that backwards direction of ends should mean negative area then $f(x) = 1 + \int_0^x f(t)\,dt$ stands here too.

3.)

We already showed that $\lim\left(1+\frac{r}{n}\right)^n = e^r$ for $r > 0$ rationals. This by monotony implies the same for the approached positive x real numbers. For negative x we need an other monotony, namely of $\left(1+\frac{1}{R}\right)^R$. We already showed this for rational R-s and it can be extended to reals. This implies that if $x_1, x_2 \ldots$ are increasing real numbers up to infinity then, $\lim\left(1+\frac{1}{x_n}\right)^{x_n} = e$. Indeed $[x_n] \le x_n \le [x_n] + 1$, so by the monotony the $\left(1+\frac{1}{x_n}\right)^{x_n}$ sequence is in between the subsequences of $\left(1+\frac{1}{n}\right)^n$.

Then for the negative case: $\lim\left(1-\frac{x}{n}\right)^n = \lim\left(\frac{1}{\frac{n}{n-x}}\right)^n = \frac{1}{\lim\left(\frac{n}{n-x}\right)^n}$ and then:

$$\lim\left(\frac{n}{n-x}\right)^n = \lim\left(1+\frac{1}{\frac{n-x}{x}}\right)^n = \lim\left\{\underbrace{\left[\left(1+\frac{1}{\frac{n-x}{x}}\right)^{\frac{n-x}{x}}\right]^x}_{\downarrow \atop e^x}\ \underbrace{\left(1+\frac{1}{\frac{n-x}{x}}\right)^x}_{\downarrow \atop 1}\right\}$$

Thus $\lim\left(1-\frac{x}{n}\right)^n = \frac{1}{e^x} = e^{-x}$. So 3.) stands for negative x values too.

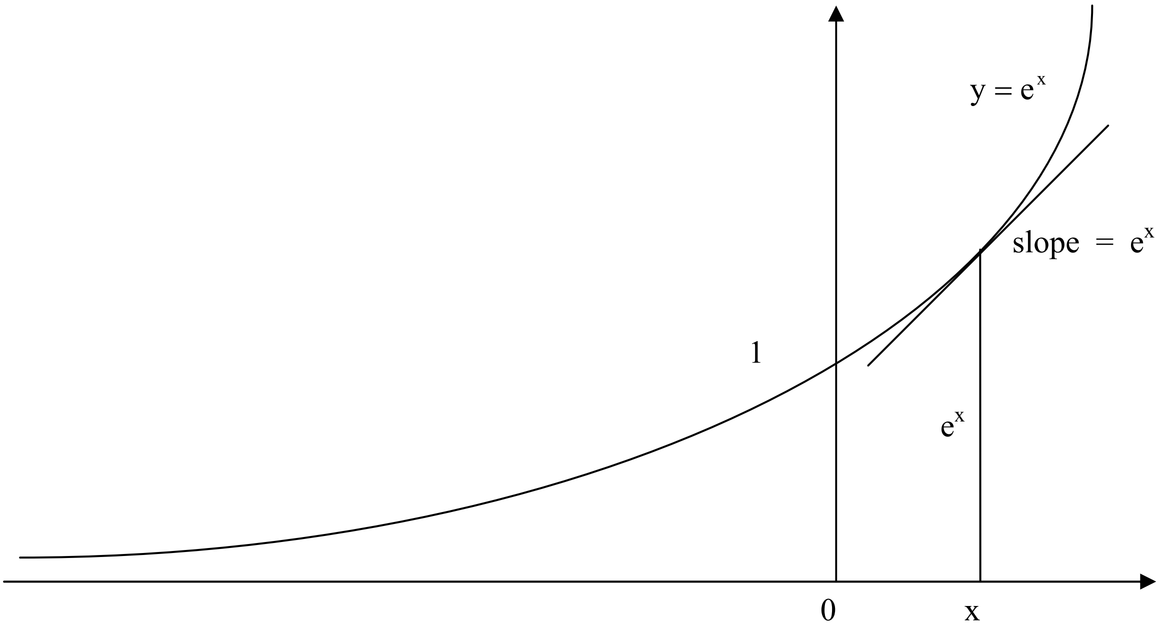
4.)

$$1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \ldots = 1 + \int_0^x \left(1 + t + \frac{t^2}{2} + \frac{t^3}{2\cdot 3} + \ldots\right) dt$$

So by 1.), 2.), 3.) this must be identical with e^x.

Our earlier proof of $e = 2 + \frac{1}{2} + \frac{1}{2\cdot 3} + \ldots$ can be easily generalized to prove 3.) but only for positive x values because as we remember it depended on the double monotony.

R The self area property of e^x by the Newton-Leibniz rule also means a self slope property. In short e^x always increases in the rate of its actual value:



R We already proved that **e** is irrational. The $\sqrt[n]{x}$ surd of an irrational x is also irrational because otherwise $x = \left(\sqrt[n]{x}\right)^n$ would be rational. So $\sqrt[n]{e}$ is irrational too!
Strangely, the seemingly easier irrationality of e^n powers is harder to prove. We'll give a complete proof for e^x with any x fraction, thus re-proving the irrationality of **e** and $\sqrt[n]{e}$.
The sum form of e^x already suggests that Lambert's first sum ratio formula will be applied but we still need some trick because in e^x all powers appear, while in Lambert's ratio the evens are in the numerator and odds are in the denominator. To separate the even and odd terms is easy though, because for e^{-x} only the odd members become negative.
This trick shows that it was logical that we waited for the general sum form of e^x and not used the easier provable positive case only.

T

1.) $$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 3\cdot 4} + \ldots$$
$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} + \ldots$$

2.) If x is rational but not 0, then e^x is irrational, in particular $e^1 = e$ is irrational.
3.) If $x > 0$ is rational but not 1, then $\ln x$ is irrational.

P

1.) Trivial.
2.) From Lambert's first sum ratio formula and our earlier result of the irrationality of the continued fraction appearing in that, the sum ratio is irrational too. Thus by 1.) :
$$\frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}}$$ is irrational and so e^x must be irrational too, otherwise $e^{-x} = \frac{1}{e^x}$ and thus this whole expression were rational too.
3.) $\ln x$ is only 0 if $x = 1$ so for all other positive values if $\ln x$ is rational then by 2.) : $e^{\ln x} = x$ is irrational.

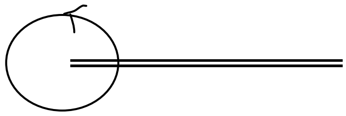
6. Trigonometric functions

D

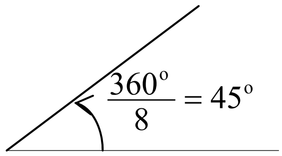
1.) The angle value of an α real number is the α° turning of a half line around its end point with the following agreements:

a.) 0° is the non turning identity of the half line:

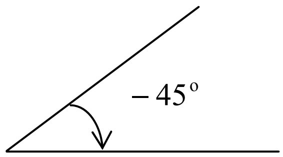
b.) 360° is the full turning in anti-clockwise direction:



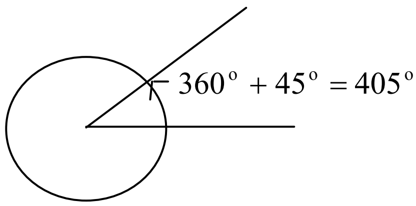
c.) All other angle value is proportional.
Example an eight of a full turn:



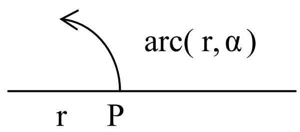
d.) The clockwise directions are for negative α values.
Example:



e.) After 360 the end results of the turnings are the same as for smaller values:



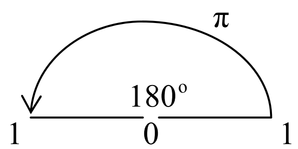
2.) The arc value of a positive real r radius and α° angle is the $\text{arc}(r,\alpha)$ real number which is the length of the turning path that the P point travels at α° turning, that was r distanced on the half line:



The area value, $\text{area}(r,\alpha)$ can be defined similarly as the swiped area.
Both arc and area has the same sign as α.

3.) The $r = 1$ fixed value gives a simple conversion of the α angle variable into $\text{arc}(1,\alpha) = \text{radian } \alpha = \text{rad } \alpha$

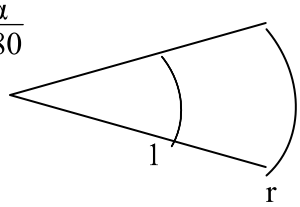
4.) $\text{arc}(1,180) = \text{rad } 180 = \pi = 3.14\ldots$



T

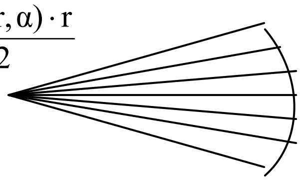
1.) $\text{arc}(1,\alpha) = \pi \frac{\alpha}{180}$ ex: $\text{arc}(1{,}360) = 2\pi$, $\text{arc}(1{,}90) = \frac{\pi}{2}$, $\text{arc}(1{,}45) = \frac{\pi}{4}$

2.) $\text{arc}(r,\alpha) = r\pi \frac{\alpha}{180}$



3.) arc(r, 360) = circumference of circle with r radius = $r\pi \frac{360}{180} = 2r\pi$

4.) $\text{area}(r,\alpha) = \frac{\text{arc}(r,\alpha)\cdot r}{2}$

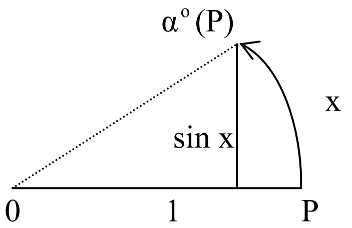


5.) area(r, 360) = area of circle with r radius = $\frac{2r\pi\, r}{2} = r^2\pi$

D

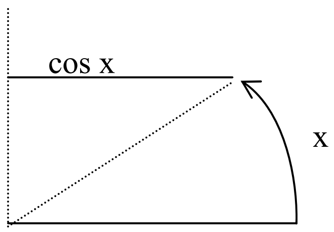
Trigonometric functions:

1.) sin x = sin rad α = sin arc(1, α) = sin α° = height of P point at r = 1 after the α° move.



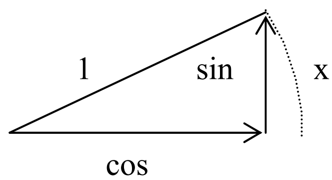
If the point moves underneath the 0 P line then the sin x is regarded as negative.

2.) cos x = cos rad α = cos α° = distance of the moved P from the vertical line through 0



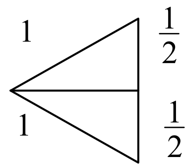
This distance could also be called the "advance" of x and when it's on the left side it is regarded as negative.

3.) $\tan x = \tan \alpha^\circ = \text{slope of } x = \frac{\sin x}{\cos x} = \frac{\text{height}}{\text{advance}}$



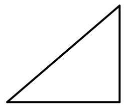
As we see we used the same symbols for the real-real functions as for the angle-real functions.
The appearing "degree" sign makes this unambiguous.

7. Complementarity and irrationality



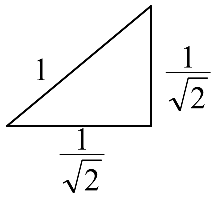
T $\sin 30^\circ = \sin\frac{\pi}{6} = \cos 60^\circ = \cos\frac{\pi}{6} = \frac{1}{2}$

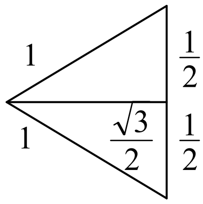
$\tan 45^\circ = \tan\frac{\pi}{4} = 1$



The Pythagoras theorem gives easy calculation of the trigonometric functions for the other cases:

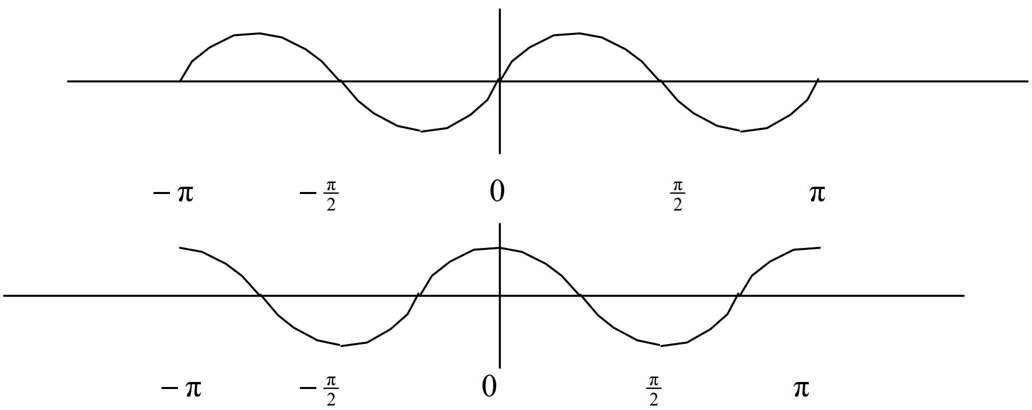
$\sin 45^\circ = \sin\frac{\pi}{4} = \cos 45^\circ = \frac{\pi}{4} = \frac{1}{\sqrt{2}}$





$\cos 30^\circ = \cos\frac{\pi}{6} = \sin 60^\circ = \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

As we see if α° and β° are two angles adding up to 90° then the sin of one is the same as the cos of the other. This is true in a more general sense: The whole sin and cos functions are simply $\frac{\pi}{4}$ shifted versions of each other:



The 90° complementarity of sin and cos, causes a much more important relation between them:

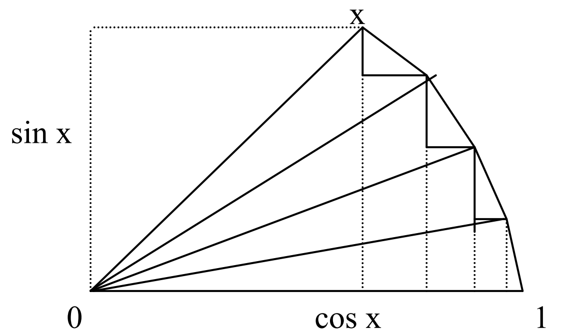
T 1.) $\int_0^x \cos t\, dt = \sin x$ and $\int_0^x \sin t\, dt = 1 - \cos x$

2.) $\sin' = \cos$ and $\cos' = -\sin$

3.) The sin and cos functions always cross the coordinate line in $\pm 45^\circ$.

P

1.)



$$\sum \frac{x}{n} \cos\, x_k = \sin x$$

$$\sum \frac{x}{n} \sin\, x_k = 1 - \cos x$$

2.) Follows from Newton-Leibniz rule
3.) Follows from 2.)

R

As we remember a strictly self area function, that is $f(x) = \int_0^x f(t)\,dt$ could only be trivial $f = 0$, while $f(x) = 1 + \int_0^x f(t)\,dt$ led to e^x. The negative case: $f(x) = 1 + \int_0^x f(t)\,dt$ leads to e^{-x} and indeed $e^x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$ so this can be verified also by integrating these powers.
This playing with the signs in the sums suggests that the relation of sin and cos areas might also be obtained from separating the even and odd terms of e^x and changing the signs.
It doesn't take a long time to figure out the perfect solution:

T

1.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

2.) If x is non zero, tan x is defined on x and non zero, then:

$$\frac{1}{\tan x} = \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots} = \frac{1}{x} - \cfrac{1}{\frac{3}{x} - \cfrac{1}{\frac{5}{x} - \ldots}}$$

3.) If x is non zero rational , tan x is defined on x and it is non zero, then it is irrational.
4.) π is irrational.
5.) If x is non zero rational then tan x is defined and is irrational.

P

1.) Easy to verify that these sums have the proper integrals or derivatives and their uniqueness can be proved similarly as for e^x. We'll also show it soon again in general.
2.) Follows from 1.) and Lambert's second sum ratio formula.
3.) From 2.) and our earlier proof of the irrationality of Lambert's c.r.s. $\frac{1}{\tan x}$ is irrational. So tan x must be too.
4.) $\tan \frac{\pi}{4} = 1$ so by 3.) $\frac{\pi}{4}$ must be irrational and so π must be too.
5.) First of all tan x is not defined only when $\cos x = 0$ that is for $x = \frac{\pi}{2} \pm k\pi$ and these are irrational by 4.). Secondly $\tan x = 0$ only when $\sin x = 0$ that is for $x = 0$ or $x = \pm k\pi$ and these are irrational again. So 3.) proves it.

8. Power sums

R The trick we applied to find the sums for sin and cos, raises the question whether there is a more systematic way. In this section we start from scratch and won't find such solution, only hints. Then in the next section we show the beautiful final result. If we start from scratch then we should drop even the factorial denominators of the powers and regard in general:

D The power sum for a $c_0, c_1, c_2 \,.\,.$ coefficient sequence is: $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \,.\,.$

R The original question was how to find for an $f(x)$ the $c_0, c_1, c_2 \,.\,.$ so that: $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \,.\,.\,.$ The reverse question, that is to find $f(x)$ for a given $c_0, c_1, c_2 \,.\,.$ is clearly less well defined because we didn't tell what forms we accept as the "found" $f(x)$. After all, $\sum c_n x^n$ is a "form" too and then there is no problem at all. There is two most basic solutions for $c_0, c_1, c_2 \,.\,.$ being all 1-s or $1, -1$ alternating.

T If $|x| < 1$ then

$$1 + x + x^2 + x^3 + \,.\,.\,. = \frac{1}{1-x} \qquad \text{and} \qquad 1 - x + x^2 - x^3 + \,.\,.\,. = \frac{1}{1+x}$$

P The identity follows from simply multiplying both sides with $1 - x$ and $1 + x$ but the left sides have a limit only when $|x| < 1$.

R From these two basic forms we can at once get new ones by either replacing x with some function of x that is by "inner" change or replacing both sides with some functions of them. For the first, a simple example is using $x := x^2$ and then:

$$1 + x^2 + x^4 + x^6 + \,.\,.\,. = \frac{1}{1-x^2} \qquad \text{and} \qquad 1 - x^2 + x^4 - x^6 + \,.\,.\,. = \frac{1}{1+x^2}$$

In fact the basic two forms can also be obtained from each other with $x := -x$ substitution. For the "outer" change we give a more interesting example: Suppose we are looking for: $\sum n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \,.\,.\,. = ?$. The following steps lead to this:

$$1 + x + x^2 + x^3 + x^4 + \,.\,.\,. = \frac{1}{1-x} \qquad / \ (\ )'$$

$$1 + 2x + 3x^2 + 4x^3 + \,.\,.\,. = \left(\frac{1}{1-x}\right)' \qquad / \ \cdot x$$

$$x + 2x^2 + 3x^3 + 4x^4 + \,.\,.\,. = x\left(\frac{1}{1-x}\right)' \qquad / \ (\ )'$$

$$1 + 4x + 9x^2 + 16x^3 + \,.\,.\,. = \left[x\left(\frac{1}{1-x}\right)'\right]' \qquad / \ \cdot x$$

$$x + 4x^2 + 9x^3 + 16x^4 + \,.\,.\,. = x\left[x\left(\frac{1}{1-x}\right)'\right]'$$

To find an even more explicit form of this we need some derivation rules:

T 1.) $[f(g)]' = f'(g)\, g'$

2.) $[f\, g]' = f'\, g + f\, g'$

P 1.)

$$\lim_{t\to x} \frac{f(g(x)) - f(g(t))}{x-t} = \lim_{t\to x}\left[\frac{f(g(x)) - f(g(t))}{g(x)-g(t)}\ \frac{g(x)-g(t)}{x-t}\right] = f'(g(x))\, g'(x)$$

2.)

$$\lim_{t\to x} \frac{f(x)\,g(x) - f(t)\,g(t)}{x-t} = \lim_{t\to x}\left[\frac{f(x)-f(t)}{x-t}\ g(x) + f(t)\ \frac{g(x)-g(t)}{x-t}\right] = f'(x)\,g(x) + f(x)\,g'(x)$$

R Now we can calculate the last result of the previous remark: $x + 4x^2 + 9x^3 + 16x^4 + \ldots =$

$$x\left[x\left(\frac{1}{1-x}\right)'\right]' = x\left[1\left(\frac{1}{1-x}\right)' + x\left(\frac{1}{1-x}\right)''\right] = x\left[\underbrace{\frac{-1}{(1-x)^2}\,(1-x)'}_{\frac{1}{(1-x)^2}} + x\left(\frac{1}{(1-x)^2}\right)'\right] =$$

$$x\left[\frac{1}{(1-x)^2} + x\,\frac{(-2)}{(1-x)^3}\,(1-x)'\right] = x\,\frac{1+x}{(1-x)^3} = \frac{x+x^2}{1-3x+3x^2-x^3}$$

And indeed:

$(x + 4x^2 + 9x^3 + 16x^4 + \ldots)(1-3x+3x^2-x^3) = x + x^2$ can be verified with term by term multiplication and a lot of work. To figure out this sum without the derivational tricks could have been much harder, so as always general principles did give an edge.
To give an even better example of this, we show two consequences of basic power sums with integration. The second will require the derivative of arc tan x, the inverse of tan x, so first we show this from the general principle:

T 1.) $(\text{arc}\, f)' = \dfrac{1}{f'(\text{arc}\, f)}$

2.) $(\text{arc}\tan x)' = \dfrac{1}{1+x^2}$

P 1.) $$\lim_{t\to x} \frac{\text{arc}\ f(x) - \text{arc}\ f(t)}{x-t} = \lim_{t\to x} \frac{1}{\dfrac{f(\text{arc}\ f(x)) - f(\text{arc}\ f(t))}{\text{arc}\ f(x) - \text{arc}\ f(t)}} = \frac{1}{f'(\text{arc}\ f(x))}$$

2.) First of all $(\tan)' =$

$$\left(\frac{\sin}{\cos}\right)' = \left(\sin\frac{1}{\cos}\right)' = \cos\frac{1}{\cos} + \sin\left(\frac{1}{\cos}\right)' = 1 - \frac{\sin}{\cos^2}\,\cos' = 1 + \frac{\sin^2}{\cos^2} = 1 + \tan^2$$

Then with 1.) $$(\text{arc}\tan)' = \frac{1}{\tan'(\text{arc}\tan)} = \frac{1}{1+[\tan(\text{arc}\tan)]^2} = \frac{1}{1+x^2}$$

T For $|x| \le 1$

1.) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \ln(1+x)$ in particular at $x = 1$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2$ This is what we called L in the Inf. Sums. Of. Recip.

2.) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \arctan x$ in particular at $x = 1$

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

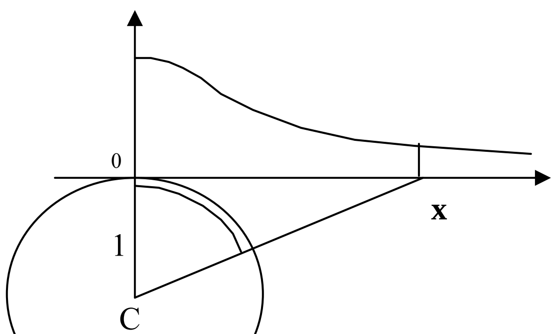
P 1.) $1 - t + t^2 - t^3 + \ldots = \frac{1}{1+t}$ $\quad / \int_0^x$

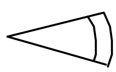
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \int_0^x \frac{1}{1+t}\,dt = \int_1^{1+x} \frac{1}{t}\,dt = \ln(1+x)$$

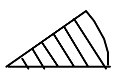
2.) $1 - t^2 + t^4 - t^6 + \ldots = \frac{1}{1+t^2}$ $\quad / \int_0^x$

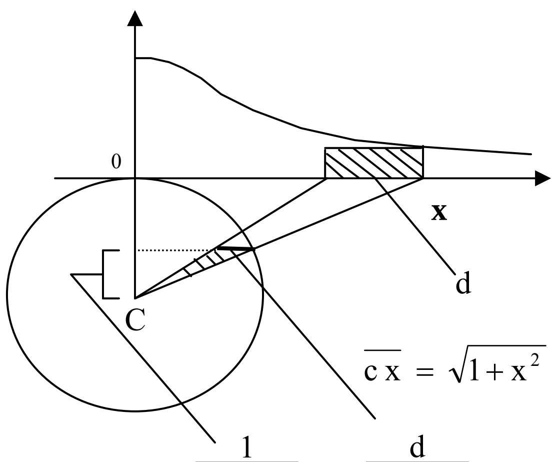
$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x (\arctan t)'\,dt = \arctan x$$

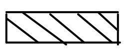
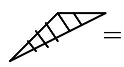
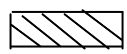
We give a direct proof for this last equality without the general inverse rule:
We'll show that the area under the $\frac{1}{1+t^2}$ function from 0 to **x** is same as arc tan x.



 $= \arctan x$

 $= \frac{\arctan x}{2}$



So  $= \frac{d}{1+x^2}$ while  $= \frac{1}{2}\frac{d}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} = \frac{1}{2}$ 

9. Generalized Newton-Leibniz rule

R Though there are no rules to find the simplest form of a power sum, we have a basic relationship between the coefficients and $f(x)$.
Unfortunately, even this is only a necessary but not sufficient condition:

T

1.) If $c_0 + c_1 x + c_2 x^2 + \ldots = f(x)$ then:

$$c_0 = f(0) \ , \ c_1 = f'(0) \ , \ c_2 = \frac{f''(0)}{2} \ , \ c_3 = \frac{f'''(0)}{2 \cdot 3} \ , \ \ldots c_n = \frac{f^{(n)}(0)}{n!} \ldots$$

2.) If f is infinitely derivable at $x = 0$, or maybe even everywhere then it’s still not necessarily true that:

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots = f(x) \text{ for all } x.$$

P

1.)

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots$$
$$f''(x) = 2c_2 + 2 \cdot 3 c_3 x + 3 \cdot 4 c_4 x^2 + \ldots$$
$$f'''(x) = 2 \cdot 3 c_3 + 2 \cdot 3 \cdot 4 c_4 x + \ldots$$

.

2.) To show a drastic counter-example we give an $f(x)$ that:
- a.) is infinitely derivable
- b.) $f(x)$ and all of its derivatives have 0 value at $x = 0$
- c.) yet $f(x)$ is not 0 anywhere else than $x = 0$

Obviously then, the sum form is identical with 0 but the function is 0 only at $x = 0$
In short: the sum only represents $f(x)$ at $x = 0$

$$f(x) = \frac{1}{e^{\frac{1}{x^2}}} \text{ if } x \neq 0 \quad \text{and} \quad f(x) = 0 \text{ if } x = 0$$

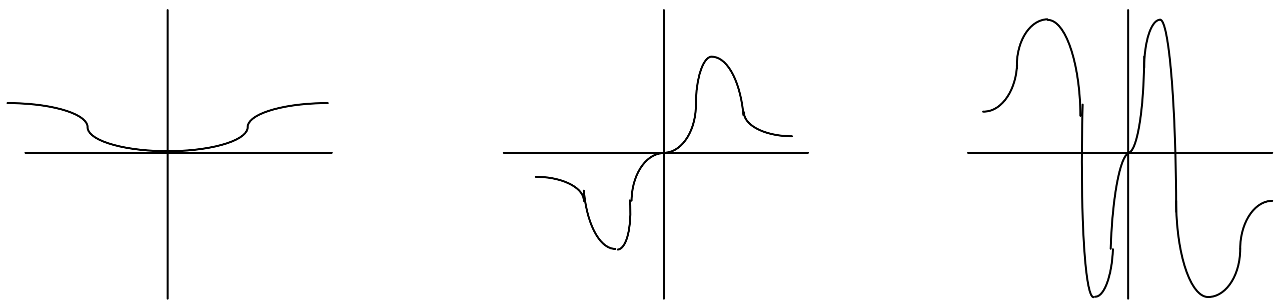
D The $f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$ sum is called the Taylor sum of $f(x)$ and of course it only exists if f is infinitely derivable at 0.
The previous theorem, thus means:

T

1.) If an $f(x)$ is a power sum then this is its Taylor sum.
2.) The Taylor sum of $f(x)$ doesn’t necessarily give $f(x)$ for all x.

R To find a condition when the Taylor sum of $f(x)$ gives $f(x)$, is to avoid similar counter-examples as we gave above. If one calculates the actual derivatives there, it turns out that they will have bigger and bigger values, in any surrounding of 0:

$f(x)$ $\qquad$ $f'(x)$ $\qquad$ $f''(x)$



So if this were always true, then functions that have limited derivatives, would be identical with their Taylor sums. Indeed this will be true even more generally, because the 0 point can be replaced by any fixed x_0. And all this is a consequence of the generalization of our most basic principle, the Newton-Leibniz rule:
The original Newton-Leibniz rule can be reformulated as follows:
If for an $f(x)$ function in the $[a,b]$ interval:

1.) $f(x)$ has its $f'(x) = \lim\limits_{t \to x} \dfrac{f(x) - f(t)}{x - t}$ derivative,
which is the local approximative change rate limit in every point.

2.) This $f'(x)$ has its $\int_a^b f'(t)\,dt = \lim \sum rect = \lim \sum out = \lim \sum in$ integral, that is the total approximative change limit on the full $[a,b]$

Then in [a,b] the actual change of $f(x)$ from an x_0 to x, that is $f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt$.

So the limits of the local and total approximations are in synchrony because the actual changes can be recovered through them. We use this assumption subjectively as we guess changes from our memory of change rates. So this law is natural, not obvious but expectable.
If we go to a higher level of abstraction and assume that the f' change rate also has an f'' change rate and so on, the repeated $f', f'', \ldots f^{(n)}, f^{(n+1)}$ derivatives exist and also that all these have integrals in the [a,b] interval then we get something totally unexpected, namely that in [a,b] any $f(x) - f(x_0)$ change can be approximated by two parts: One main part that only depends on $x - x_0$ and the $f', f'', \ldots f^{(n)}$ derivatives at x_0, and a smaller part that is a fraction of $\int_{x_0}^{x} f^{(n+1)}(x-t)^n\,dt$. Going even further in abstraction, if the derivatives exist up to infinity then the total $f(x) - f(x_0)$ change can be calculated from $x - x_0$ and the sequence of derivatives at x_0. Thus our seemingly easy jump from f' to the existence of all higher derivatives must be something very strong because it implies that these derivative values at any x_0 place determine all other $f(x)$ values.
In a primitive analogy this reminds us of the holistic feature of nature, namely that every place contains the full, so to say "the ocean in a drop".
But staying within mathematics, we can tell exactly why we don't feel this extremely strong condition when we "easily say" that f has the repeated derivatives or in short "infinitely derivable". The reason is that since derivability means a "nice" tangent at every point of the function we feel that the derivative function is also "nice". This is not so, that is a derivative can be very "ugly". Thus every additional condition of the next derivative is a brand new strong restriction on the previous derivative and thus on the original f too.

T Generalized Newton-Leibniz rule

1.) Lemma: Let f abbreviate $f(t)$

$$\left[f + \frac{f'}{1}(x-t) + \frac{f''}{2}(x-t)^2 + \ldots + \frac{f^{(n)}}{n!}(x-t)^n \right]' = \frac{f^{(n+1)}}{n!}(x-t)^n$$

2.) Finite generalization: $f(x) = f(x_0) +$

$$f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$$

3.) Infinite generalization: If in [a,b]:
   a.) f is infinitely derivable,
   b.) all $f^{(n)}$ derivatives have their integrals,
   c.) there is a B bound so that for all derivatives: $\left| f^{(n)}(x) \right| < B$.

   Then $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \ldots$

P

1.)

$[f]' = f'$

$[f + f' \cdot (x-t)]' = \cancel{f'} + f'' \cdot (x-t) + \cancel{f' \cdot (-1)}$

$\left[f + f' \cdot (x-t) + \frac{f''}{2}(x-t)^2\right]' = \cancel{f' + f'' \cdot (x-t)} + f' \cdot (-1) + \frac{f'''}{2}(x-t)^2 + \cancel{f'' \cdot (x-t)(-1)}$

and so on

2.)

$$\int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt = \left[f(t) + f'(t)(x-t) + \ldots + \frac{f^{(n)}}{n!}(x-t)^n \right]_{x_0}^{x} =$$

$$f(x) - f(x_0) - f'(x_0)(x-x_0) - \ldots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

3.)

$$\left| \int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt \right| < \frac{B}{n!}\int_{x_0}^{x}(x-t)^n \, dt = \frac{B}{n!}\left[-\frac{(x-t)^{n+1}}{n+1}\right]_{x_0}^{x} = B\,\frac{(x-x_0)^{n+1}}{(n+1)!} \to 0$$

With $a = \left| x - x_0 \right|$, all we have to show is that $\frac{a^n}{n!} \to 0$ for any $a > 0$.

First of all: If $n > a - 1$ that is $n + 1 > a$ then $\frac{a^{n+1}}{(n+1)!} = \frac{a}{n+1}\frac{a^n}{n!} < \frac{a^n}{n!}$.

Thus our sequence is monotone decreasing after a. Secondly: If $n > k\,a$ that is $n \geq [k\,a] + 1$

then: $\frac{a^n}{n!} = \frac{a^{[a]}}{[a]!}\,\frac{a}{[a]+1}\,\frac{a}{[a]+2}\cdots\frac{a}{[k\,a]+1}\cdots\frac{a}{n} < \frac{a^{[a]}}{[a]!}\,\frac{1}{k}$ is arbitrary small.

$$\frac{a}{[a]+1} \wedge 1, \quad \frac{a}{[a]+2} \wedge 1, \quad \frac{a}{[k\,a]+1} \wedge \frac{a}{k\,a} = \frac{1}{k}, \quad \frac{a}{n} \wedge 1$$