Introducing Growth

Let f be an arbitrary function and S be an arbitrary set.

The f widening of S is the S^f set defined as S itself if f is not defined on S and being $S \cup \{ f(S) \} = S + f(S)$ if f is defined on S.

If $S^f = S$ then we call S an f terminating set. This can mean two things by our definition, namely either that f is not defined on S or that $f(S) \in S$.

The union widening of a B set is $\bigcup B = \{ e ; \exists S (e \in S \in B)\}$ which is the set that contains all those e elements that are elements in any S elements of B and thus is the combining of all elements of B.

Let s be an arbitrary set that we will regard as a starting element.
The starting or first stage is {s}. The second stage is $\{s\}^f = \{s\} + f\{s\} = \{ s , f\{s\}\}$ if it is different from {s} that is if {s} was not terminating, that is if $f\{s\} \neq s$.
The third stage is $(\{s\}^f)^f = \{ s , f\{s\} , f\{ s , f\{s\}\}\}$ if it is different from the second. That is if $f\{ s , f\{s\} \} \neq s$ nor f {s}.
Similarly we define the wider fourth and so on stages.
The first infinite stage comes about by a union widening of all the earlier stages, that is as:
$\{ s , f\{s\} , f\{ s , f\{s\}\} , f\{ s , f\{s\} , f\{ s , f\{s\}\}\} \; . \; . \; . \; \}$
This can again be f widened if it is not a terminating stage.

We call a widening a set of the stages up to any point.
So the first widening is {{s}} containing only the {s} stage.
The second widening is $\{ \{s\} , \{ s , f\{s\}\} \}$ containing the sates {s} and { s , f {s}}.
The third widening is $\{ \{s\} , \{ s , f\{s\}\} , \{ s , f\{s\} , f\{ s , f\{s\}\}\}\}$ containing the stages {s} , { s , f {s}} and { s , f {s} , f { s , f {s}}} .
The first infinite widening is:
$\{ \{s\} , \{ s , f\{s\}\} , \{ s , f\{s\} , f\{s , f\{s\}\}\} , \; . \; . \; . \; \}$
This is “open” meaning that it has no maximal element that is stage in it.
To make it “closed” is easy by adding the first infinite stage to it:

$$\{ \{s\} , \{ s , f\{s\}\} , \{ s , f\{s\} , f\{s , f\{s\}\}\} , \; . \; . \; . \; , \{ s , f\{s\} , f\{ s , f\{s\}\} , . \, . \} \}$$

In general, this new combined element is $\bigcup B$ for any open beginning.
The “growth of s through f ” is the first f terminating stage.
The G “growth of f from s ” is the widening up to the first f terminating stage.
The beginnings in G are easy to describe. They are B subsets so that every S stage in B must be subset of every other T in the G – B = E end part of G:
$(S \in B \text{ and } T \in E) \rightarrow S \subset T$
This simple definition is the reason that it’s better to target G than the growth of s.
But what’s even better is that the growth of s comes out of G explicitly anyway.
First of all, we don’t have to restrict our union combing claim to open beginnings because if a beginning is closed, that is has a widest stage than the union is that anyway.
Secondly, the above definition of B allowed B to be empty or the full G. The empty has to be excluded for the union claim but the full G can be allowed and so $\bigcup G$ is the growth of s.
Our first rule says the start, the second the f widening up to the first f termination, the third the union widening, the fourth the f termination for the widest stage of G.

1. $\{s\} \in G$

2. $(S \in G \text{ and } S \neq \bigcup G) \rightarrow (S^f \neq S \text{ and } S^f \in G)$

3. B is a non empty beginning of G $\rightarrow$ $\bigcup B \in G$

4. $(\bigcup G)^f = \bigcup G$

A simple sign of our failure is that we can not even prove that $\{\{s\}\}$ is a beginning.
We know that this has only one stage $\{s\}$ and this stage has no real subset and so all other stages should be in the end, but we can not prove that all other stages contain $\{s\}$.
This relates to a deeper problem. Namely that all our rules only tell what must be in G but don't really exclude possible junks. But this is still not all. An other problem is that we can not prove the existence of any G even with junks because we have no easy example that satisfies our rules. We simply aimed to high. So quite strangely our rules are too weak because allow junk and at the same time too strong because require the final f termination.
The solution is to describe not G rather the partial W widenings. This can be achieved by simply dropping 4. and replacing it with a rule that would avoid junk:

1. $\{s\} \in W$

2. $(S \in W \text{ and } S \neq \bigcup W) \rightarrow (S^f \neq S \text{ and } S^f \in W)$

3. B is a non empty beginning of W $\rightarrow$ $\bigcup B \in W$

4. $(S , T \in W \text{ and } S \neq T) \rightarrow (S \subset T \text{ or } T \subset S)$

As immediate success, now we can see that our description is not hollow because $\{\{s\}\}$ is a simple example of a W. Secondly we can also prove that this is a B beginning in all W.
Indeed, since $\{s\}$ has no other subsets, all stages must contain it and so the beginning end relation is true. We could now show that the second expected beginning $\{\{s\} , \{ s , f \{s\}\}\}$ is indeed a beginning too but we should instead realize the more general fact that:
For any B beginning that is not the full W, the $\bigcup B$ or $(\bigcup B)^f$ stage is always a new stage not in B rather in the end and this is a subset of all other stages in the end. Indeed:

If B is an open beginning then by 3. $\bigcup B \in W - B$.
We claim that all other T stages in the $W - B$ end are wider.
By 4. it's enough to show that $T \subset \bigcup B$ is impossible.
Every $e \in \bigcup B$ is coming from an S with $S \subset T$ so $e \in T$ too and thus $\bigcup B \subseteq T$.

If B is a closed beginning then $\bigcup B \in B$ and this is the widest stage in B.
If $B \neq W$ then also $\bigcup B \neq \bigcup W$ so by 2. $(\bigcup B)^f$ is wider than $\bigcup B$ so $\in W - B$.
We claim that all other T stages in the $W - B$ end are wider.
By 4. it's enough to show that $T \subset (\bigcup B)^f$ is impossible. But $\bigcup B \subset T$ and so T would be in-between $\bigcup B$ and $(\bigcup B)^f$ which is impossible because $(\bigcup B)^f$ has only one extra element.

So we proved that the non empty ends have $\bigcup B$ or $(\bigcup B)^f$ as narrowest stage in them.
And so the non empty ends always do have a narrowest minimal element. An other way of saying this is that the common part of the elements in the end is itself an element in the end.

This common part can be defined as: $\bigcap E = \{ s ;\ \forall S (S \in E \rightarrow s \in S) \}$
And so our result is that: E is a non empty end of W $\rightarrow$ $\bigcap E \in E$
This seems similar to the third rule but we claim common part instead of combining and most importantly we claim this to be not merely in W rather in E itself!
To see the importance of this rule, we generalize it to arbitrary C collection of stages:

$(C \subseteq W$ and C is non empty $) \rightarrow \bigcap C \in C$

Indeed, lets regard all those S stages that are subsets of all stages in C. This is a B beginning because any stage that is not in B must contain some stage in C and so it can not be subset of any stage in B. So we have a minimal stage in the W– B end and it has to be a C member otherwise it were a subset of all C members and so it would have to be in B.
An other way of saying this rule is that for any kind of stages there is a narrowest such kind.
So then if we had junk among our stages then there would have to be a narrowest junk J.
The impossibility of this follows from the fact that for any S stage in a W, the stages that are subsets of S, are a B beginning of W and S is $\bigcup B$ or $(\bigcup B)^f$.
So the sub stages of our previous J were a B too and J itself is $\bigcup B$ or $(\bigcup B)^f$.
The B contains no junk and thus these too are non junk either, rather the must.
Of course, this was a bit of external logic and not quite exact.

The exact road can not talk about the possible junks rather has to produce the original goal, the G growth of f from s and thereby the growth of s through f that is $\bigcup G$.
And now the heuristic idea is to get G as the combining of all possible W-s except that we have to add the combining of this combining as a possibly new maximal element too:

$G = \bigcup Y + \bigcup \bigcup Y$ where $Y = \{ W ; 1.$ and 2. and 3. and 4. $\}$

Then the first part is to show that this G will obey the rules too, so G is a W.
Then the fact that $\bigcup G$ is an f terminating one is easy.
Indeed, G has to be the widest possible W but if $\bigcup G$ is not f terminating that is $(\bigcup G)^f \neq \bigcup G$ then $G + (\bigcup G)^f$ were a wider W.

So $\bigcup G$ is terminating, that is the original 4. applies for G.

The inheritances of the new rules to the combined G rely on two simple facts:

Any closed B beginning in a W, is itself a W'.

For any two W , W' widenings, one is beginning of the other.

To prove the first is quite boring by checking all the four rules to remain for B.
The proof of the second on the other hand is very tricky.
We simply regard those beginnings of W and W' that are definitely common. There have to be some because {s} is one such for example. Next we combine these common beginnings. This total B set will be beginning in both sets.
Indeed, for every S in B and T outside B we must have $S \subset T$ because S was already in a beginning and T outside of it. So this B is also the widest possible common beginning.
If B is not proper in one of them that is the full W or W' then of course we are finished.
But this has to be the case because if B were proper beginning that is leaving non empty ends in both W and W' then $\bigcup B$ or $(\bigcup B)^f$ were in these ends and adding it to B would make a wider common beginning.

Now the inheritances are easy:

1. is trivial because {s} is in all widenings and we added $\bigcup\bigcup Y$ to $\bigcup Y$, so $\bigcup G$ will definitely be in G.

2. could be said also as: $S , T \in W$ and $S \subset T \rightarrow S^f \neq S$ and $S^f \in W$.
So both 2. and 4. are trivial because any two S , T in G came from W , W' and by the second fact above, one contains the other as beginning so S , T both were in that wider one obeying 2. and 4. already.

3. requires both facts.
If B is the full G then again our adding of $\bigcup\bigcup Y$ to $\bigcup Y$ will guarantee it at once.
If B is not the full G and so the G – B = E end is non empty, then there is a T in it and it had to come from some W. Enough to show that B was a beginning in W already because then rule 3. was true in this W making it true in G too.
For B being a beginning in W it is enough if it was merely a subset in W. Indeed, then already can not be stage outside of B in W that were not wider than all B elements because otherwise such would be in G too refuting that B is a beginning. So, finally enough to show that any S stage in G that is subset of T was already in W. Now, S had to come from some W' and in that the stages up to S including S , are a closed beginning and so by the first fact form a W''. Then by the second fact this W'' has to be a beginning of W.

So the end result is that for any f and s we have an G that obeys all rules.
This can also be called as a "wellordering" of $\bigcup G$ if by this we mean an ordering that has a next after any beginning. The real point was of course to achieve this as a well defined set by collection rather than relying on our intuitive growth in time.

This was the strictly technical part of the wellordering.
The next step is to apply this for any A set.
This simply means to obtain the arbitrary A set as a $\bigcup G$ and this trick is heuristic too.
To get G we have to find some f , s and this is only where the Axiom Of Choice enters.
We regard a c choice function that picks an element from all subsets of A including A itself.
In fact, the choice from A that is c (A) is our s and then the rest of c will give our f as:
$f(S) = c(A - S)$.
The beauty of this is that it is always outside S so f termination means being not defined but it is always defined except for S = A. So we indeed get that $\bigcup G = A$.
This obtained wellordering of sets can then be used to compare different arbitrary A , A' sets.
Indeed they are always beginnings of each other too and thus one is equivalent to a subset of the other a side result. This makes Cantor's definition usable to all sets.
These consequential points overshadowed the technical beauty that actually hides the concept of growth or rather how to avoid it. Beside this, an other internal didactical error used in the technical part is the abstract handling of wellordering. The beginnings having a next element was what I claimed. And indeed any ordered set when cut into the beginning and end section gives four simple choices. Having a last in the beginning or not and having a first in the end or not. The beginning always having a next is the same as the end always having a first. As we saw, this generalizes to the fact that all subsets have a first.
An other, seemingly opposite direction is not to generalize from ends to arbitrary subsets rather just to some very special subsets. Namely, to say that there shouldn't be any backward infinite sequence : $\ldots , s_3 , s_2 , s_1$. This is just one particular subset without first but to exclude this at once excludes all others too. Indeed, if there is any subset without first then it means that picking any s_1 element from the subset, we must still have more before s_1. So we can pick again one s_2 and this can not be the first so again we have earlier ones. And so on we obtain the backward infinite sequence.

In fact, we used the Axiom Of Choice tacitly but more importantly, this impossibility of a backwards infinite sequence is a paradox. Indeed, we have infinite many elements forward but miraculously we can not pick infinite many backwards. This paradox is a bit tamed by realizing that even though we have infinite many natural numbers, picking a number from this infinity at once reduces to possible earlier ones to finite many. Wellordering makes this same reduction to finiteness but even for sets that actually can be arbitrary big.

There is an opposite direction of abstractness that targets not the particular wellordering meaning rather the heuristic method of set combining as next widening. Again, we can generalize from beginnings to arbitrary C subsets but now this is more educational and gives true directions of the future details. This is symbolically reflected by revealing that here C should stand for not collection, rather "cofinal".
A C subset in a W can be two kind relative to a B beginning:
Namely, whether C will contain wider stage than all the stages in B or not.
If not then we again have a second duality whether B contains stage that is wider than all C elements. If again the case is not, then actually C is going all the way inside B.
It might skip some stages but it surpasses every member yet doesn't surpass them all with a single stage. This is the meaning of the mentioned cofinality.
Amazingly, in this case it is evident that $\bigcup B = \bigcup C$, the combined sets of them are the same.
Indeed, any e element that appears in any element of B will eventually be collected by a wider set in C. So, requiring that the total of this C is in W means not more than the envisioned full B sets combined and added.
The other two cases when C went too far or not far enough, can also be justified by seeing that there are B beginnings that go exactly as far as C.