Contents

Mathematical Logic

1. Similarity With And Difference From Natural Languages

In normal written languages, we have an alphabet, words and sentences. This three leveledness will be in Mathematical Logic too. In fact, even to the point, that the first two levels give the specifics of what we talk about, while the third is merely the general rules. Indeed, the words are all different, but the rules of forming sentences, the grammar is common for all words. So then the words would correspond to the different mathematical fields, while grammar to Mathematical Logic.
The first two levels however, have a certain oppositeness. In real languages, the alphabet is a small, finite set, usually twenty to thirty letters, while the vocabulary is practically infinite, if we allow more and more new words. In math, an opposite approach is taken, the set of letters is infinite, while the word formations are only finite. This formal difference is understandable much better, after revealing the most important fundamental similarity, between natural and mathematical languages. This fundamental similarity concerns the intended meanings of a language. Indeed, all languages talk about something outside the language itself, namely either in the real world, or in our imagination. Both in math and in the everyday fields, these two mix up. We may look at figures we draw on a black board, but we don't use the actual physicality of those pictures, rather just to help our imagination. Similarly, even if we write a novel, we use our memories from the real world. So what is this fundamental common feature, in the meanings of all languages? It is a simple duality of the meanings, that we can call <u>object</u> and <u>state</u>.
Objects or things, simply are as they are, while states are always true or false.
The states are above the objects, they are about the objects.
A state can be about one single object, like a book being red, or Peter running. Other states can be about more objects, like a book being on the table, or Peter and Paul being brothers. The simple state of one object could also be called a <u>property</u>, while a very complicated state among many objects as a <u>situation</u>. In normal languages, this distinction of object and state is not emphasized, because we use more detailed classification as nouns, verbs, adjectives and so on. By understanding that object versus state, is the real duality, we can even attack the conventional grammar. Indeed, why is running a verb, while happy an adjective? One might say that verbs are actions we do by our own will, while adjectives are merely features we possess. But this logic is not correct, for example, we say "be happy,", so you can voluntarily be happy? Or maybe the "be happy," only means you should be happy. But then again, if someone is an "evil" person, is that not voluntarily? If so, then why are we despising him? So as we see, real language, is just trying to sneak out of the boring simplicity of objects and states, while we have to embrace this simplicity.
Then we can return to the oppositeness of alphabet and vocabulary in the common and mathematical languages. In normal languages, the objects are referred to by words, while in mathematics, by letters. Of course, it's not quite true, for the very reason that normal languages must use words, namely that simply there are not enough letters. But in math, we can overcome this problem by using subscripts or special symbols. So, $x_1 , x_2 , x_3 , \ldots\ldots , x_n$ are regarded as new letters. In fact, we could even use infinite real numbers as subscripts, like $x_\pi = x_{3.1428\ldots.}$. In a sense, these are words, but we regard them as letters, so the words could mean something completely new. This new is the formation of states or new objects. Since letters only refer to objects, thus to get states, we definitely need such words, but to form new objects, is not absolutely necessary. If we do have such word formed objects, then they are called secondary objects, as opposed to the primary alphabetical or lettered objects. As we said, usually there can be infinite many primary objects, but there are only finite many word formations. So does that mean that we'll only have finite many states and secondary objects? No! The explanation is the crucial new feature of our mathematical words, that they are not fix combination of letters, rather "frames" into which, any letters can be written. Thus, of course, even just one word formation or frame, can produce infinite many states or objects. Only the number of the allowed letters are fixed for a frame, while all possible letters can be used. Best to imagine this, as a bracket, having a fix number of commas, separating the possible letters: (, ,) .

Here we have two commas, which of course means three possible letters. This then, would give a different state or object for all possible letterings, like (a , b , c) or (x , y , z) or (1 , 2 , 3) and so on. We can use [ ] for another word formation or { } for another, but we don't have enough type of brackets, so a more practical way is to use letters or even english words in front of the brackets:

A (x , y , z) , r (a , b , c) , Brothers (a , b , c) , sum (x , y , z) . . . .

Sometimes we can use special symbols instead of the commas to distinguish different brackets:

(x < y) , (x + y) , . . . We can even use only these special symbols without brackets at all.

The object versus state difference is so fundamental, that we'd expect at least mathematicians to use totally different symbols for them. Unfortunately, it is not so. We'll try to do our best, at least here within Mathematical Logic, so small letters will always refer to objects, while capital ones or capital initial ones to states. Like above, we used Brother since it's a state, while sum since it's an object.

There is an other important special feature of mathematical languages, which only becomes useful after we go to the third level that is logic. This is the splitting of the letters that is the symbols for primary objects into names and variables. We'll assume that every object has a unique name in fact mathematics only deals with these names. The physical objects are not for math. This also means that the states as reality are regarded as merely collections of those names that are in states. But math will talk also about how states relate to each other. There we need variables that can stand for any objects. This includes also the possibility that two different variables represent the same object. At first this may seem strange. Indeed just by looking at x and y, why should they be the same? But then we soon realize that in states like x sees y that is See (x,y), it is an unavoidable coincidence that someone looks in a mirror and then x=y. But this raises the similar question about names: Should we allow there too to have two different ones for a single object? Definitely not! Our whole idea is to use the names instead of the objects so they must represent each other uniquely! Still it may happen that we want to refer to single objects temporarily. Then these temporary names or as we'll call them, <u>constants</u> can be different and still mean the same object or rather unknown name. The expression "constant" is logical if we think of them as special variables with non varying that is fix meaning.

As we mentioned, the third level above the letters and the words, is the sentence and the formation of this, is in both normal and mathematical languages, actually above the specific meanings. That is, giving the grammar in normal language, while the logic in math.

Just as in grammar, we have special words to form sentences, we have special frames to form logical states. In fact, they correspond exactly to everyday meanings.

2. The Five Frames Of Logic, Grammatics

$\neg$ = not = the only single state frame

$$\left.\begin{array}{l}\wedge \;=\; \text{and} \\ \vee \;=\; \text{or}\end{array}\right\} \text{ frames for two states } = \text{ logical operations}$$

$$\left.\begin{array}{l}\forall\, x \;=\; \text{every} \;=\; \text{all} \\ \exists\, x \;=\; \text{there is} \;=\; \text{some}\end{array}\right\} \text{ frames for one state with a selected letter } = \text{ quantors}$$

Examples:
$\neg\,(3 < 5)$ = 3 is not smaller than 5 = false

$(3 < 5) \vee (5 < 1)$ = 3 smaller than 5 or 5 smaller than 1 = true

$\forall x\ (1 < x)$ = Every number is bigger than 1 = false

Both operations and quantors can be expressed with each other using “not”:

$A \vee B = \neg\,(\neg A \wedge \neg B)$

$A \wedge B = \neg\,(\neg A \vee \neg B)$

$\exists x\ A = \neg\, \forall x\ \neg A$

$\forall x\ A = \neg\, \exists x\ \neg A$

Other frequently used logical operations can be expressed with $\wedge$ and $\vee$ and so, by the aboves, even with single operations:

$A \nabla B$ = either A or B = $(A \vee B) \wedge \neg\,(A \wedge B)$

$A \rightarrow B$ = A implies B = if A then B = $\neg\,(A \wedge \neg B)$ = $\neg A \vee B$

$A \leftrightarrow B$ = A iff B = If A then and only then B = $(A \rightarrow B) \wedge (B \rightarrow A)$

Older versions of mathematical logic preferred the $\rightarrow$ implication as a basic operation, but today all modern proof theoretical approaches realize that the true nature of mathematical logic is best expressed by starting with $\wedge$ and $\vee$. The implication is also very confusing to start with, because it gives the false impression of casual relationship. The true nature of $\rightarrow$ is exactly how we replaced it by $\wedge$ or $\vee$. Indeed, $A \rightarrow B$ merely means that its impossible that A is true and B is false, or in other words, A must be false or B must be true. The followings help:

$(2 \times 2 = 5) \rightarrow$ the pope is a woman, is true, because the first part was already false.
$(2 \times 2 = 5) \rightarrow$ the pope lives in Rome, is true again, because the first part is false again.
$(2 \times 2 = 4) \rightarrow$ the pope is a woman, is false because the first part is true, yet the second is false.
$(2 \times 2 = 4) \rightarrow$ the pope lives in Rome, is true again because both parts are true.

The false impression about implication is also helped by math itself. Indeed in mathematics, we use the implication to get new theorems from old ones. In fact, the oldest logical rule was “modus ponens” claiming the obvious that if A is true and $A \rightarrow B$ is also true, then B is true too.
Of course, just as well, we could use A and $\neg A \vee B$ to get B.

The two quantors are the real heart of mathematical logic! Amazingly, already Aristotle's formal logic used them. Unfortunately, he didn't realize the duality of objects and states and thus, the "every" and "there is" were used only to combine states. This then lead to the totally superficial "syllogisms" like, "every monkey is an animal, and all animals die, so every monkey dies". The recognition of objects came actually quite late in mathematics by the introduction of variables and parameters in equations. It's strange why geometry didn't supply enough stimulus to discover the need for lettering objects. After all, when lines cross, they have common points. Here the lines can be objects that relate, but also the points within the lines. Probably this was too much to be formalized at once for Euclid.

When we use a quantor in front of a state, we have to specify which object we mean to be "every" or "there is". This of course, makes those objects completely disappear from the state. After using $\forall x$ or $\exists x$ the x is not an object anymore. In a sense, it would be much more logical to replace x by some special letters kept especially for quantification. We could even avoid letters at all and put the quantors themselves in place of x. Then of course, we would have to use subscripted quantors for the different letters. For example,

$\forall x \exists y \exists z \forall u (x < y \wedge z < v \wedge z < u \wedge x < v \wedge y < w)$ could be written as:

$$\forall_1 \exists_1 \exists_2 \forall_2 (\forall_1 < \exists_1 \wedge \exists_2 < v \wedge \exists_2 < \forall_2 \wedge \forall_1 < v \wedge \exists_1 < w)$$

In fact, we could use subscripts that give the order of all quantors and then we wouldn't even need them to be used in front of states, because their order is determined:

$$\forall_1 \exists_2 \exists_3 \forall_4 (\forall_1 < \exists_2 \wedge \exists_3 < v \wedge \exists_3 < \forall_4 \wedge \forall_1 < v \wedge \exists_2 < w) =$$
$$\forall_1 < \exists_2 \wedge \exists_3 < v \wedge \exists_3 < \forall_4 \wedge \forall_1 < v \wedge \exists_2 < w$$

I don't think that this system will ever "catch on", because the old fashioned lettered use is already accepted. The main thing to understand is that once a letter is quantized, then that letter can be replaced by any new letter unused in that state. On the other hand, in different quantized states we can use the same letters for quantification, if we want to. This choice of letterings seems like a mere technical detail, but it can be used to achieve meaningful results. Namely, Herbrand who was an early pioneer of these investigations used the changing of quantor variables to replace logical rules.

If we allow the five logical symbols to be used in any order, then we obtain the <u>loose</u> states of the language. There are two special states:

If we don't use quantors, only $\neg$, $\wedge$, $\vee$, then the obtained states are called <u>situations</u>.

This makes sense, because in these the letters are really talking about objects themselves.

If we don't have any variables or the ones we have are all quantized so they don't mean objects any more, then the state is a <u>statement</u>. This makes sense again because these states can not vary according to the names we imagine in place of the variables, so they are simply true or false.

So to obtain a statement from a state we have to put names in place of some variables, that is concretize them and quantize all the other variables.

The two special states can be the same, namely a situation statement is using only names as letters and not using quantors at all. These could also be called as concrete statements.

Without any logical rules, that is just by relying on our intuitions, merely by introducing the quantors into everyday language, we get an amazing clarification of what we mean. We could keep the conventional system of using words for objects, but also incorporate the frames. Such mixture of everyday and mathematical language could be called "grammatics" and should be taught by elementary english teachers before more involved math, that is alongside of simple arithmetics. This not only would help with grammar and composition, but later would make math much easier.

I tried to convince the education department in Hungary many decades ago about this idea, of course without any success. I also tried it out with my daughter, Timi (Timea), with great success, when she was eight years old. In the end she became a lawyer, so after all I failed. Recently, I found the few examples she transformed to grammatics with her beautiful handwriting. So now I present these, to convince you of my point:

1.) Timi's bicycle has been stolen. =

They stole Timi's bicycle. =

Somebody stole Timi's bicycle.

$S\ (x, y) = x$ steals y
t_b = Timi's bicycle

$\exists x\ S(x, t_b)$

But this can be further refined by instead of t_b, using:

$P(x, y) = x$ possesses y

$B(y) = y$ is a bicycle

t = Timi

$\exists x\ \exists y\ [S(x, y) \wedge P(t, y) \wedge B(y)]$

2.) Children like candy.

$Ch(x) = x$ is a child

$L(x, y) = x$ likes y

$C(y) = y$ is a candy

$\forall x\ \forall y\ [(Ch(x) \wedge C(y)) \rightarrow L(x, y)]$

3.) If someone looked around on Mars, he'd see a strange view.

$L(x) = x$ looks around

$St(y) = y$ is strange

$S(x, y) = x$ sees the y view

$P(x, y) = x$ is on planet y

m = Mars

$\forall x\ \exists y\ \{[P(x, m) \wedge L(x)] \rightarrow [S(x, y) \wedge St(y)]\}$

4.) Every woman has a moment in her life,
When she'd like to do that's not alright.

This sentence was from an old Hungarian song, before the second world war. My dear analysis teacher Czach wrote it on the black board when he started his lectures in first year. He said if somebody can't tell what the negative of the sentence is, then he or she won't be able to understand what is convergence and divergence. He was right, and also way ahead of his time! Mathematical Logic is still last years subject at all universities in the world!

$W(x) = x$ is woman

$A(y) = y$ is alright to do

$L(x, y, z) = x$ would like to do y at z

$$\forall x \left\{ W(x) \rightarrow \exists z \exists y \left[L(x, y, z) \wedge \neg A(y) \right] \right\}$$

Lets see the negative:

$$\neg \forall x \left\{ W(x) \rightarrow \ldots . \right\} = \exists x \neg \left\{ W(x) \rightarrow \ldots . \right\} =$$

$$\exists x \neg \neg \left\{ W(x) \wedge \neg \exists z \exists y \left[L(x, y, z) \wedge \neg A(y) \right] \right\} =$$

$$\exists x \left\{ W(x) \wedge \forall z \forall y \neg \left[L(x, y, z) \wedge \neg A(y) \right] \right\} =$$

$$\exists x \left\{ W(x) \wedge \forall z \forall y \left[L(x, y, z) \rightarrow A(y) \right] \right\}$$

There is a woman so that if she would like to do anything anytime, then it is alright.
It's not as poetic as the original positive was.

3.Quantification Of Reality

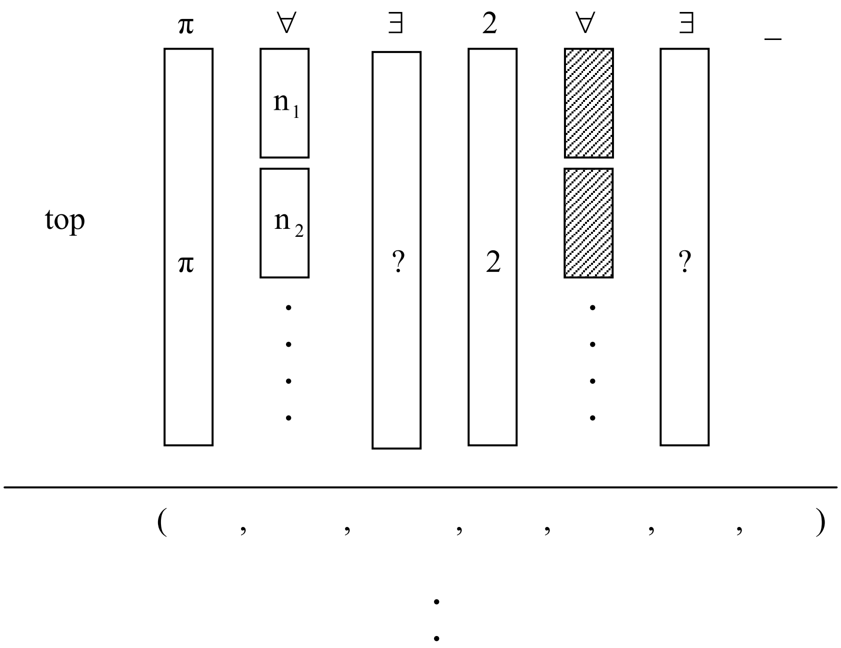
As we said Math is only concerned with the names but not with the real objects. So in short the names are the mathematical reality. But this is empty, after all the names are just a set of symbols. We have to tell what is the reality of the states too! We also said that logic will establish the relationships among the states that's why we need variables. Still a direct reality of the states would be useful as a guide to develop logic. This direct reality is amazingly simple. All we have to do is tell exactly what names are in what states. Or in a better organized way, for every state we need a list of those groups of names that are in that state. A group of k names is also called a k-tuple if we put them in a bracket like: $(n_1, n_2, \ldots, n_k)$.
So a list of such k-tuples is the reality of this k-participant state.

$$\begin{array}{c} (\;,\;,\;,\;,\;.\;.\;.\;.\;.\;,\;,\;) \\ (\;,\;,\;,\;,\;.\;.\;.\;.\;.\;,\;,\;) \\ \vdots \end{array}$$

The problem with this visualization of reality is that it shows much more than what we wanted.
Indeed we only wanted to tell what tuples are "in" but now they are given in a particular order too! This is an unfortunate fact but we have to pretend to ignore this order. In fact we might have so many tuples that can not be listed at all. So our "list" is merely an alternative for collection, that is merely having the tuples in a basket if you wish. The real advantage of visualizing the tuples as a list, especially under each other, is that this way the same participants are under each other as columns.
To see the columns is vital in how the quantors $\forall$, $\exists$ and concretization by names like π , 2 can have a meaning. Formally, we can use them simply by writing each above a column:

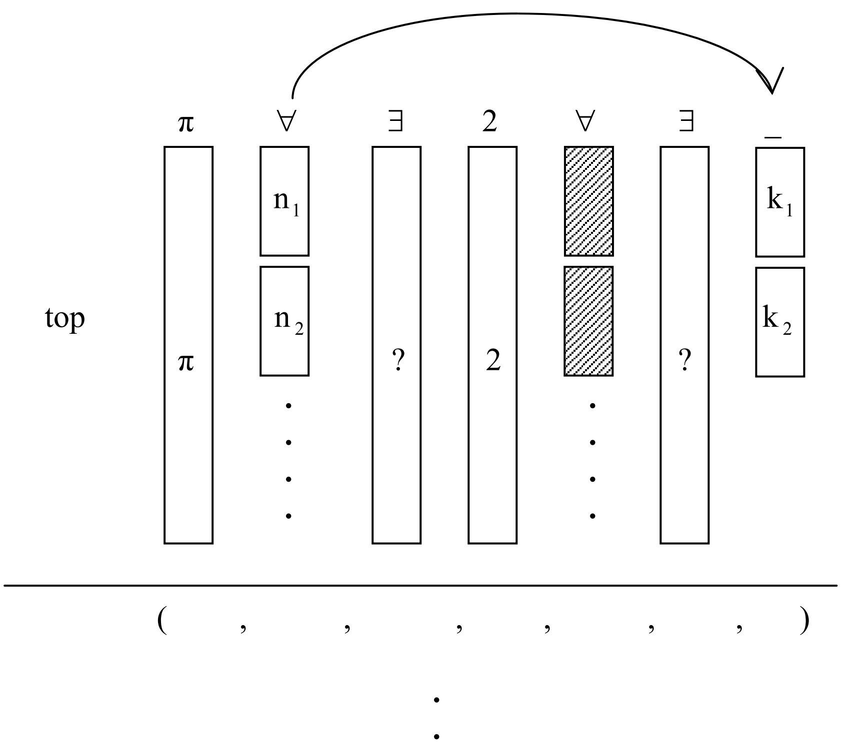
$$\begin{array}{ccccccccccccccc} & \pi & & \forall & & \exists & & 2 & & \forall & & \exists & & \_ & \\ (& & , & & , & & , & & , & & , & & , & &) \\ (& & , & & , & & , & & , & & , & & , & &) \\ & & & & & & & \vdots & & & & & & & \end{array}$$

But what should this mean? π , 2 , $\exists$ are not obvious at all. π might mean that there is a π under it somewhere, but also that only π appears. $\exists$ is even more confusing, because if it means what we think it should, namely that there is something there, then it is obvious. Only $\forall$ seems to be clear by its meaning as "every", thus, requiring that in that column, all names appear. Yet, even $\forall$ will mean more, because if there are more of them, then we'll require not only that separately in these columns, all names appear, rather that in those columns all combination of names appear in some line. In our example above, it means that all possible pairs of names must appear in those two columns in a line. Now the concretization and $\exists$ can be defined by requiring even more from the $\forall$ combination. Namely, that all such combination appear, even in lines having the concretized name, in its column and some fix, but not concretized name, in the $\exists$ columns. To see more clearly our definitions, we can bring up some lines to the top of our list. Namely, lines with the concretized names under those and fixed arbitrary ones, under the $\exists$. We can do that, with having all combinations to appear under the $\forall$. The combinations of the $\forall$ columns can be achieved by simply keeping one fix, and then going through all names for another. Then fixing a new name again, and taking all combinations under a third $\forall$ and so on. So if we use empty blocks to denote same elements, we have a list like:



Under the second ∀, I even put blocks shaded, denoting that their not same elements, rather all possible names appear.

This top of the list, expresses what we claim by our definitions, but it is not unique. First of all, the ∃ columns, that is the mystery names may have different choices, but most importantly, the unmarked columns can be different too. To see why, we have to remember, that we only claimed that all ∀ combinations appear, but one combination might appear many times, so our "topping" is just one possible choice from those repeats. The choice of a combination means bringing up that line to the top and that brings the last unmarked element in it. So in the last unmarked column, mixed and unrelating names appear, in addition, they are dependent on how we chose our topping for the unique ∀ combinations. In fact, lucky top choices might show some rules in the unmarked columns. In our example, in the single last one, such rule could be if there is a partial repetition in that column, namely following the sub blocks of the ∀ column. Above, there was only two ∀, so the only possibility could be that the last column repeats exactly as the first ∀.



As I said, the blockings of the $\forall$ were up to us. We could have fixed the second $\forall$ column elements, and go through all of them in the first repeatedly. Then we couldn't spot the weirdness in the last column, because the repeating elements were not under each other directly. The arrow I drew from the first $\forall$ to the unmarked column represents the dependence. If we have more than two $\forall$ and more unmarked columns, we can have more complicated dependences like:

$$\_ \quad \forall \quad \exists \quad \forall \quad \pi \quad \forall \quad \forall \quad \_ \quad \exists \quad \forall \quad \_$$

Here all unmarked columns are dependent in a very entangled way, but the meaning is completely clear. There is a topping in which: The first column has the same name whenever the sixth has the same. The eighth unmarked column has the same whenever the fourth and sixth $\forall$ have the same pairs. And so on. A very special dependence is the following:

$$\pi \quad \forall_1 \quad \__1 \quad 2 \quad \__2 \quad \forall_2 \quad \forall_3 \quad \_ \quad \exists \quad \forall_4 \quad \__3$$

It's hard to see why it is special, but observe:

$\__2$ depends on $\forall_2$

$\__1$ depends on $\forall_2$ and $\forall_1$

$\__3$ depends on $\forall_2$ and $\forall_1$ and $\forall_4$

So the dependence is widening. This also means that we can rearrange the columns so that we don't need arrows, because every unmarked column will depend on the earlier (left) $\forall$ columns:

$$\pi \quad 2 \quad \exists \quad \forall_2 \quad \__2 \quad \forall_1 \quad \__1 \quad \forall_4 \quad \__3 \quad \forall_3 \quad \_$$

We also brought the concretized 2 column to the beginning, which was just to "clear the field". More importantly, we had to take the undepending third $\forall$ and _ to the end. Indeed, if an unmarked column is not dependent, then it merely means its dependent on all $\forall$ columns, because one combination of those gives a particular value of the unmarked one in a chosen topping. Finally, we can realize that bringing the $\exists$ in front made also sense because it is fixed independently of any $\forall$ column. So then we might as well use new $\exists$ symbols for the unmarked columns as well, because they mean fixed values too. So the final quantification of the list is:

$$\pi \quad 2 \quad \exists \quad \forall \quad \exists \quad \forall \quad \exists \quad \forall \quad \exists \quad \forall \quad \exists$$

This form is exactly what we meant by statements, in our earlier language of logic, so its not surprising that it gives the exact meaning, when reading from left to right:

We can bring lines to the top, so that among these, π and 2 are fixed. Also, some fixed name can be under the first $\exists$. Then, all names appear under $\forall$ and for each of these, there is a fix under the second $\exists$. Then with every combination of the already mentioned columns, all names appear under the next $\forall$. And so on. This formalism even washes away the simple fact, that all $\forall$-combinations appear. All this for the sake of simplifying the dependences. And of course, it all only works for widening dependences. Luckily, nothing forbids us to use more quantifications for a same list, so with different widening dependences, we can describe non widening ones too. But now, I'll show what is the real and amazing advantage of this widening or consecutive quantification. This will prove that it is objectively special, not merely convenient for us to abbreviate or to read:
One would think, that a quantification is a very special claim about a list. In other words, only very rarely, could a randomly chosen list satisfy a pre chosen quantification. If we only allow consecutive quantification, it still leaves the restriction very small. If further more, we also allow concretizations, then it increases the restriction. And yet, I'll show that such statements about a list are not rare at all.
The clue lies in the fact, that we used the word statement. So obviously, something will be related to our earlier language of logic. The two quantors are already used, $\wedge$ and $\vee$ would be hard to use for one list, so clearly the missing $\neg$ is the solution. Indeed, a list contains some tuples of names, so the opposite of a list is simply all those tuples that can be formed from our names but are not in our list. This opposite list is also called the complementer of the original. But that's not enough! We need the opposite of a quantification too. We already defined those by naïve meanings as:

$$\neg \forall x\ A \ = \ \exists x\ \neg A \quad , \quad \neg \exists x\ A \ = \ \forall x\ \neg A$$

Using this repeatedly means simply changing all $\forall$ to $\exists$ and all $\exists$ to $\forall$. If we call such altered quantifications as complementers as well, then our claim is simple:
If for a list, a quantification is not true, then the complementer quantification, must be true for the complementer list!
So for example, if our above $\pi\ 2\ \exists\ \forall\ \exists\ \forall\ \exists\ \forall\ \exists\ \forall\ \exists$ quantification is not true for our list of tuples, then $\pi\ 2\ \forall\ \exists\ \forall\ \exists\ \forall\ \exists\ \forall\ \exists\ \forall$ must be true for the list containing all other tuples. Already the first two concretization seems unbelievably strong, and most strangely, they remain in the complementer the same. So what if there are no π or 2 in those columns at all? Then the topping is impossible, or rather empty, so nothing can be true about it. Exactly! But then, a lot of π and 2 must appear in the complementer list, namely every combination of the other names will appear with fixed π and 2. So then, of course, every quantification is true about that list. Similarly, lets see the first $\exists$! If it was a phony claim originally, because lets say there wasn't even one name in that third column that appears with all names in the fourth column, then that exactly means that all names appear in the third column in the complementer list. Indeed, if an n name were missing there, then all combinations containing n were in the original list contradicting, that there wasn't one with all fourth column combinations.

4. Minimal Reality

If some quantification are true about a list then obviously they are true about a topping of the list, after all that's how we define its truth. The question is whether a topping can be reduced further, and the answer is beautifully simple. Yes, in fact every topping has a special set of lines that are already enough. In other words, deleting all other lines, the quantification would remain true. What's more, this set of lines has to be in any subset of the topping, that would satisfy our quantification.
So in short:
There is a minimal subset of a topping, that is still good for the same quantification.
But there is more:
1. This minimal subset is truly just a sequence of lines.
2. It has to be chosen only by the $\forall$ columns, and it will automatically keep the dependences.
This also means that we go back to our roots and don't fall under the spell of formalism to use statements. So we use less and yet prove more, because any complicated dependence, that is arrowed quantification, is automatically guaranteed. Lets remember, that a topping was chosen so, that all given names or primary, that is original $\exists$-s have a fix value, with which all $\forall$ combinations appear exactly once. The unmarked or unrestricted columns were then specified if they had special repeatances following certain $\forall$ columns. But now, we don't have to worry about this, for the simple reason that if such simple dependence exists, then it remains for any subset of the topping.
Indeed, for example:

$$\forall \qquad \forall \qquad \_ \qquad \forall$$

meant that in lines, where those three columns under the $\forall$ have a certain combination, then under _ also the same name will appear. Now clearly, this remains true for any subset of lines. So all we have to guarantee is the $\forall_1, \forall_2, \ldots, \forall_q$ columns producing all q-combinations.

The start is very easy! In a topping, we have the $g_1, \ldots, g_m$ given names and the $h_1, \ldots, h_k$ hidden names that were claimed by the primary $\exists$-s. These two, the g-s and the h-s don't determine the particular topping, because there could have been more same $\forall$ combinations. And so, we made arbitrary choices to collect an all q-combination top. Our choices of top, for a given $g_1, \ldots, g_m, h_1, \ldots, h_k$ is not visible in the $\forall$ columns, after all we made sure to collect all q-combinations. The difference is in the unmarked leftover columns. These are the ones about which, we showed above that we don't have to worry, because they will keep all their inner features. All we need is to reduce the $\forall$ columns. If we just leave out one line of $\forall$ combinations, then all those names that appear in it, must be eradicated completely, otherwise this combination will be missing. But if we leave out those names, that means the deletion of new lines and thus, new names must be eradicated again. In the end we may have nothing. We have to go by building up what we keep:
$g_1, \ldots, g_m, h_1, \ldots, h_k$ obviously must be among our names, so we should make all q-combinations out of these. Then, we can start with those lines, where under $\forall_1, \ldots, \forall_q$ all the q-combinations from g, h appear. Unfortunately, these lines will contain new $n_1, \ldots, n_p$ names in the unmarked columns. So, $n_1, \ldots, n_p$ must be added to our $g_1, \ldots, g_m, h_1, \ldots, h_k$ list of unavoidable names. Then we can make all q-combinations from $g_1, \ldots, g_m, h_1, \ldots, h_k, n_1, \ldots, n_p$ and finding these under the $\forall$-s will tell which new lines to keep. This of course brings in new $n_{p+1}, \ldots, n_r$ names, and so on, we never finish! But that's alright! This never finishing infinite sequence of names together will perfectly do! Indeed, any combination from this sequence is already at a stage that we finished. So under the $\forall$-s, looking up the line where they match, the line was already taken, so the names in the unmarked columns were taken too. So our list is complete. The minimality of this list is obvious too. But only for our chosen topping. Our sequence is not even the minimal for the given $g_1, \ldots, g_m, h_1, \ldots, h_k$, because these can have other topping and then the first $n_1, \ldots, n_p$ can already be different.

There is a little mistake we made repeatedly in our proof. Right at the beginning the $g_1, \dots, g_m, h_1, \dots, h_k$ list assumes that all h-s are different from each other and the g-s. It doesn't have to be so, but if it isn't we can just omit the repeats. Then again $n_1, \dots, n_p$ can have earlier names, which we just omit and so on. This just rectified mistake is directly related to our next question: Instead of finding some minimal list for a quantification inside an already existing list, can we create one out of nothing? Of course, we wouldn't completely create it out of nothing, because if our quantification has the $g_1, \dots, g_m$ given names, then we can start with these. Then of course, our primary $\exists$-s need new $h_1, \dots, h_k$ names. Forming all q-combinations for the $\forall_1, \dots, \forall_q$ columns from $g_1, \dots, g_m, h_1, \dots, h_k$ we can introduce new $n_1, \dots, n_p$ for each unmarked columns. And so on, we obviously repeat the whole previous proof!

Unfortunately, we are wrong! When we had a list to start with, then the dependences automatically inherited to our minimal sub sequence. But now all we get is a sanitized empty sequence. In fact, we see now that the rectification above that allowed the "collapse" of some new elements to old ones, was the thing that made our new name choices not entirely new, rather satisfying the dependences, while here we had no such guide. So we'll modify our construction for consecutively quantized cases, that is for statements.

We can start as above, but after $g_1, \dots, g_m, h_1, \dots, h_k$, not all the $\forall$-columns are regarded, only the first group of q_1 many $\forall \dots \forall$. So q_1-combinations are formed from $g_1, \dots, g_m, h_1, \dots, h_k$ and then for the next group of q_2 many $\exists \dots \exists$, we choose q_2-tuplets of names for each q_1-combination of the old names. This leads to $n_1, \dots, n_{p_1}$ new names. Then, from $g_1, \dots, g_m, h_1, \dots, h_k, n_1, \dots, n_{p_1}$ we again have to form combinations, but not just for the next q_3 many $\forall \dots \forall$ quantors, but also for the old q_1 many $\forall \dots \forall$. So all possible q_1 and q_3 combinations are created and for each, a new q_4-tuple of names for the q_4 many $\exists \dots \exists$. This continues our names up to n_{p_2}. Then of course, all q_1, q_3, q_5 combinations are formed and to each new q_6-tuplets of names, and so on.

In the end, we get a sequence of names that is not only good because all $\forall$-columns have every combinations, but also the widening dependence is satisfied.

Instead of going in tuples of combinations, and new namings, we can go strictly quantor by quantor, but still achieve the group alternations. This will be useful for later, so here we go:

5. Namings

A quantification is an $\exists$-quantification if it starts with $\exists$ and $\forall$-quantification, if starts with $\forall$.
A case of a quantification is a new one, obtained from it by changing the first quantor to a naming.
Of course if this was the only quantor then it becomes a concrete name tuple in full.
I call a set of quantifications $\exists$-named if every $\exists$-quantification has a case in it.
I call a set of quantifications $\forall$-named if every $\forall$-quantification has all cases in it.
Here "all cases" means using all names that are used in the set of quantifications.
I call a set of quantifications named, if it is both $\exists$-named and $\forall$-named.
Observe that being $\exists$-named or $\forall$-named means not just the first quantors having cases, because the cases themselves can start with the same $\exists$ or $\forall$ quantor again.
But being cased, that is for both quantors means even much more, because then every statement must have totally quantorless full namings with the proper meanings of the quantors.
And yet, singular "namings" can achive this.
An $\exists$-naming of a set of quantifications is adding a case of a $\exists$-quantification, with a new name.
An $\forall$-naming of a set of quantifications is adding a case of an $\forall$-quantification, with an old name.
Now the method to get a list for a statement is simple!
We'll regard the statement as a starting set and widen it to a named one, by namings.
Suppose the statement starts with $\exists$. Use $\exists$-namings till the set becomes $\exists$-named.
Then use $\forall$-namings till it's $\forall$-named, then again $\exists$-namings, and so on, alternatively.
If the statement starts with $\forall$ then of course we start with $\forall$-naming.
As we proceed, fully named cases, that is, concrete name tuples are formed too and the total of these gives the list. This is so because the full list of statements will be named.
The alternating use of $\exists$ and $\forall$ namings is not quite similar. Indeed, $\exists$-namings only have to use the previous $\forall$-namings, but the new $\forall$-namings have to go back to all earlier $\exists$-namings and use the new names.
As an example, lets regard the $1\ \forall\ \exists\ \exists\ \forall$ statement, where 1 is the only given name and we keep on using the natural numbers as new names too.

1	$\forall$	$\exists$	$\exists$	$\forall$	
1	1	$\exists$	$\exists$	$\forall$	$\forall$-namings = using 1
1	1	2	$\exists$	$\forall$	$\exists$-namings = making new names from 2 , 3 , . . .
1	1	2	3	$\forall$	
1	2	$\exists$	$\exists$	$\forall$	$\forall$-namings = using 2 , 3
1	3	$\exists$	$\exists$	$\forall$	
1	1	2	3	1	
1	1	2	3	2	
1	1	2	3	3	
1	2	4	$\exists$	$\forall$	$\exists$-namings = making new names from 4 , 5 , . . .
1	3	5	$\exists$	$\forall$	
1	2	4	6	$\forall$	
1	3	5	7	$\forall$	

1	4	∃	∃	∀
1	5	∃	∃	∀
1	6	∃	∃	∀
1	7	∃	∃	∀
1	1	2	3	4
1	1	2	3	5
1	1	2	3	6
1	1	2	3	7
1	2	4	6	1
1	2	4	6	2
1	2	4	6	3
1	2	4	6	4
1	2	4	6	5
1	2	4	6	6
1	2	4	6	7
1	3	5	7	1
1	3	5	7	2
1	3	5	7	3
1	3	5	7	4
1	3	5	7	5
1	3	5	7	6
1	3	5	7	7

∀-namings = using 4 , 5 , 6 , 7

1	4	8	∃	∀
1	5	9	∃	∀
1	6	10	∃	∀
1	7	11	∃	∀

.
.
.

∃-namings = making new names from 8 , 9 , . . .

The concrete name tuples, which are number tuples above, will be a sublist in our list, and that sublist is a perfect list for our quantification. This seems quite convincing from the method itself. To show it in general, first seems quite problematic though, because we don't have an exact definition of what such satisfying or validity should mean. A best solution could be to define it for longer and longer quantifications, that end up with the one we use at the top of the list.
Indeed, quantorless or concrete name tuples are "valid" in a list if they are simply in the list.
One quantored ones are valid according to their quantor meaning. Two quantored ones, by the first quantor's meaning, reducded to one quantored valid ones, and so on. So this is an exact opposite of the case dequantifications one by one, which go from the outside. Then it's easy to see that our method indeed creates a list in which all cases and thus the original quantification too, is valid.
The method of singular namings was defined for sets of quantifications. This was used in our method, because the single quantification we started with, widened to sets. But we could start already with a whole set of quantifications, namely as a simple case with a sequence of quantifications.
Imagine them, one by one, on a line. Clearly, our method could be used separately for each, but this would be useless. First of all, we would have to use separate $1 , 2 , 3 , \ldots$ sequences of names, and worst of all, the total of these wouldn't be correct for all quantifications.
Amazingly, we can use the common names $1 , 2 , 3 , \ldots$ for all the quantifications as follows:
We do two alternations under the first, then start the second with one. Then, add one more to both, and start the third one. Add one again to all, and start the fourth, and so on. All columns will be alternated upto infinity. Of course, the namings are created continually from the same $1 , 2 , 3 , \ldots$ and all earlier ones are used in all columns in the $\forall$ phase.
This fact, that infinite many quantifications can be valid over $1 , 2 , 3 , \ldots$ will mean later that all of our theories have such artificial reality in a single sequence as universe.
Kronecker, the arch enemy of Cantor, said that God created the natural numbers, and all others are made up by man. Strangely, the aboves prove that he was right in this weird sense too about realities.

6. Situation Matrix

We want to continue our approach from reality towards logic, but also introduce the earlier mentioned logic operations of $\wedge$, $\vee$ and $\neg$. We call the usage of all the five logical symbols in arbitrary order as a loose build up. Now we show that there is a very easy way to turn any loosely built state into a very strict one.
First of all, all the $\neg$-s can be moved deeper and deeper inside the state until they only appear in front of basic states of the particular language. So seemingly, $\neg$ can be eliminated from logic.

Moving $\neg$ in:

$$\neg \neg A = A$$
$$\neg \forall x A = \exists x \neg A$$
$$\neg \exists x A = \forall x \neg A$$
$$\neg (A \wedge B) = \neg A \vee \neg B$$
$$\neg (A \vee B) = \neg A \wedge \neg B$$

The quantors can be moved oppositely, that is outward until they all stand in front of the state.

If x is not appearing in B:

$$\forall x A \wedge B = \forall x (A \vee B)$$
$$\forall x A \vee B = \forall x (A \vee B)$$
$$\exists x A \wedge B = \exists x (A \vee B)$$
$$\exists x A \vee B = \exists x (A \vee B)$$

If x does appear in B, then simply change x for any new letter!
After bringing out all the quantors, we'll only have $\wedge$ and $\vee$ combinations inside and amazingly these can be reduced to a two leveled combination by:

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$
$$A \wedge B = B \wedge A$$
$$A \vee B = B \vee A$$

The first two rules can move $\wedge$ and $\vee$ into a second member. The last two allows the same for first members too. Thus, using these, we can move them till we end up with two possible final forms:

$$(A_1 \wedge A_2 \wedge \ldots) \vee (B_1 \wedge B_2 \wedge \ldots) \vee \ldots$$
$$(A_1 \vee A_2 \vee \ldots) \wedge (B_1 \vee B_2 \vee \ldots) \wedge \ldots$$

Here, all the A , B –s are basic states or their negative ($\neg$).
Both of these final forms can be changed into the other. For example, for just two members:

$$(A \wedge B) \vee (C \wedge D) = \left[(A \wedge B) \vee C \right] \wedge \left[(A \wedge B) \vee D \right] =$$

$$(A \vee C) \wedge (B \vee C) \wedge (A \vee D) \wedge (B \vee D)$$

$$(A \vee B) \wedge (C \vee D) = \left[(A \vee B) \wedge C \right] \vee \left[(A \vee B) \wedge D \right] =$$

$$(A \wedge C) \vee (B \wedge C) \vee (A \wedge D) \vee (B \wedge D)$$

The final two general forms can be abbreviated as:

$$\begin{bmatrix} A_1 & A_2 & \dots \\ B_1 & B_2 & \dots \\ \cdot & \cdot & \\ \cdot & \cdot & \end{bmatrix} \quad \text{or} \quad \left[\begin{array}{c|c|c} A_1 & A_2 & \dots \\ B_1 & B_2 & \dots \\ \cdot & \cdot & \\ \cdot & \cdot & \end{array}\right]$$

These are called situation matrixes.
In the first, the lines represent $\wedge$-s and the different lines are connected by $\vee$-s.
In the second, the columns are meant as $\vee$-s and the vertical lines as $\wedge$-s .
We'll only use the first form.
The lines, that is $\wedge$-s of basic or negated basic states will also be called as scenarios.
This makes sense in two ways, namely as scenarios of basic states that are in them, but also as different possible scenarios as "or " choices of the whole situation matrix.
Indeed, a matrix is true if at least one line, that is scenario of it is true.
Using the matrixes, every state or statement can be written as quantors in front of such:

$$\forall x \; \exists y \dots \begin{bmatrix} \quad & \quad \\ & \end{bmatrix}$$

To do a transformation from a loose form to this strict matrix form is the best exercise for a layman, to understand the basics of logic. Lets see an example:

Euclid proved that there are infinite many prime numbers. The simplest way to say this within a stricter number theory is claiming that for every number, there is a bigger number that is prime.
The trivial products are : $1 \bullet y = y$ or $y \bullet 1 = y$.
A non trivial product is not containing 1, that is $u \bullet v = y$, with neither u or v being 1.
This of course means both u and v being bigger than 1.
The composite numbers are such non trivial product values, "composed" from two bigger than 1 numbers.
A prime is simply a non composite, that is a number that can not be composed.
That would make 1 a trivial prime which is usually excluded.
The reason for this is to make the prime factorizations unique.
For the infinity of primes of course this exclusion is immaterial, so we allow 1 as prime.
We'll need two basic relations:

$x < y$: x is smaller than y

$u \bullet v = y$: u times v is y

Euclid's claim is:

$$\forall\, x\ \exists\, y\ \{ x < y \wedge y \text{ is prime} \}$$

$$\forall\, x\ \exists\, y\ \{ x < y \wedge y \text{ is not composite} \}$$

$$\forall\, x\ \exists\, y\ \{ x<y \wedge \neg \exists\, u\, v\ (u \bullet v = y \wedge 1 < u \wedge 1 < v) \}$$

$$\forall\, x\ \exists\, y\ \{ x<y \wedge \forall\, u\, v\ \neg\ (u \bullet v = y \wedge 1 < u \wedge 1 < v) \}$$

$$\forall\, x\ \exists\, y\ \{ x<y \wedge \forall\, u\, v\ (u \bullet v \neq y \vee 1 \nless u \vee 1 \nless v) \}$$

$$\forall\, x\ \exists\, y\ \forall\, u\, v\ \{ x<y \wedge (u \bullet v \neq y \vee 1 \nless u \vee 1 \nless v) \}$$

$$\forall\, x\ \exists\, y\ \forall\, u\, v\ \{ (x<y \wedge u \bullet v \neq y) \vee (x<y \wedge 1 \nless u) \vee (x<y \wedge 1 \nless v) \}$$

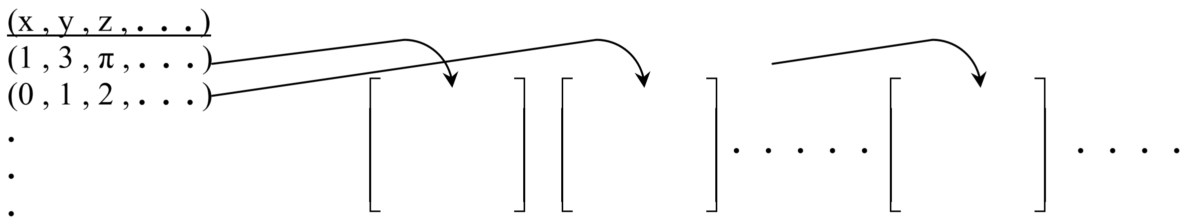
$$\forall\, x\ \exists\, y\ \forall\, u\, v \qquad \begin{bmatrix} x<y & , & u \bullet v \neq y \\ x<y & , & 1 \nless u \\ x<y & , & 1 \nless v \end{bmatrix}$$

The original intuitive meaning that for any x there is a bigger y prime, is represented when in this matrix, only the first line or scenario is true. The other two lines are the "fine print", the crucial details of excluding the trivial products.

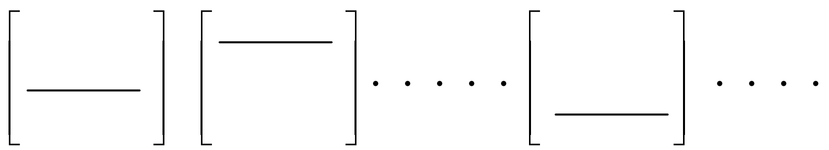
7. Realization of a Matrix

Let an M matrix contain m variables x, y, z, . . . and the appearing basic states in it, be A, B, C, . . . If we know the realities of A, B, C, . . . then it's quite easy to find the reality of M too. Indeed, all we have to do is collect the names in the realities of A, B, C, . . . Then, form all m-tuples from these names, we can even imagine them under each other. Then, check line by line that if we write them into A, B, C, . . . whether M becomes true or false, and only keep the lines where M is true.
An important detail was left out, namely that we have to be able to identify which column goes to which variable. So, we should write x, y, z, . . . above the columns. The order is not important, after all we start with the full list of all m-combinations anyway.
The reverse question would be that if we start with a (x , y , z , . . .) m-tupled reality for M, then how to find realities for each A, B, C , . . . so that they produce the one we started with.
Obviously, this reverse problem is much harder, in fact we should rather ask the question if such individual realities for A, B, C, . . . exist at all. We might even think that even this reversed "possibility" question of realities from M to A, B, C, . . . is so complicated, that we can't have a simple vision of it. We are wrong. We got overwhelmed with the complicatedness because we didn't really follow on with the original direct reality creation for M. We just said, check the lines where M is true. But when is M true? Well, that's very simple, when at least one line is true. After all, that's what all that the lines as "or-s" or scenarios mean. In fact, a line is an "and" of its members, so even more specifically, we need at least one line with everything in it to be true. Then the reversal can be visualized quite well. Indeed, imagine we repeat our matrix infinitely, and now not under each other, like lines of a reality, rather after each other from left to right. Then take the assumed reality of M, and write its lines into the M-s one by one.

Reality of M   Matrixes of M



Thus all the M matrixes become different by being filled with the different name combinations.
Then in order to be true, each must have at least one line to be completely true. So the possible truth, that is the simple possibility of M's list of reality, means that we can pick a line from each matrix and going through this selection of lines, the whole infinite "line" contains only true basic states or negatives:



The problem was to find basic states that make such infinite line true. Thus, first it might seem that any line choices can determine a choices for the basic states. Not so, because we not only have basic states, but also their negatives. So, in one infinite line, we probably end up having A and ¬ A as well, and then of course, we can't choose both to be true. Of course, A and ¬ A wouldn't be in itself a problem, the real problem is A (1, π, 3,) and ¬ A (1, π, 3,) , that is, opposite basic states with same namings, because then we don't know whether (1, π, 3,) should be in A-s list or not, that is, to be in ¬ A-s list. Now the line selection is showing its real importance, namely to avoid such negative pairs. This also shows, that even though we got a simple visual meaning of the reversal of a reality. To find such could be very hard, indeed, it requires to check all possible line choices, from infinite many matrixes, and find one without any opposite pairs, with same namings.

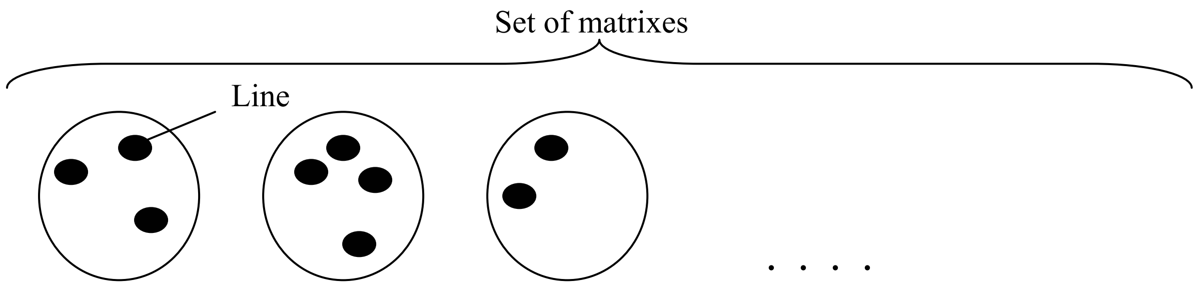
The most important fact on which the whole Mathematical Logic rests is the following:
We don't have to check all infinite many matrixes to know if there is a successful selection of lines!
It's enough to find non contradictory lines from any finite set, chosen from the set of all matrixes.
So, the choice of lines, from all matrixes is broken into two choices:
Choose any finite subset of matrixes and then choose lines from this.
This also sounds like a certain loss of concreteness, because we merely claimed that if for every finite set of matrixes, the line choices are possible, then there is one for all the matrixes.
The proof of our claim though, will be constructive in some sense. Not in the sense that the finite possible choices would directly give the total, but in a weaker sense, that we can make special choice of the finite subsets, that will create a total. Unfortunately, these third choices of finite subsets will not be concrete. So, just to make it clear why we have three choices:

1. The full choice of lines from all matrixes is the goal.
2. The choice of lines from any finite matrixes is already enough to guarantee a total.
3. The choice of some specially arranged finite sets will be used to create the total.

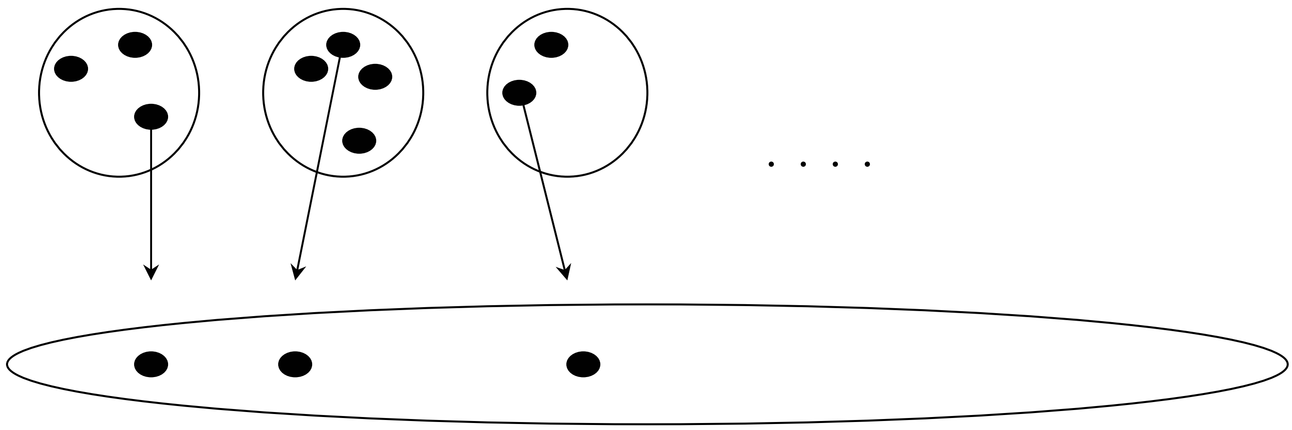
A much better concreteness appears regardless of the proof, just by the simple restating our claim into its negative form. Indeed, if all finite choices guarantee a total, then if there is no total choice, then there must be a finite set without possible choices too. If we add what choice means, that is one without contradiction, then the concreteness I claim comes out even more:
If all choices from the total are contradictory, then there is a finite set of matrixes, where already all choices are contradictory. But in a finite set of matrixes, we can easily go through all possible line choices and verify one by one, the contradictions.
So, if we change our goal from possibility to impossibility, then we are in a much better position!
Indeed, finding a finite set of matrixes that is impossible, that is leads to contradiction with all line choices clearly means that the total is impossible. But what's more, our claim also shows that this finding is always possible if the total of matrixes is impossible. This is actually where logic is going! Giving simpler and simpler ways to detect impossibility. This might sound strange now, but a heuristic idea will make it clear.

8. Taboo Avoidance

We might think that our big claim about the matrixes is somehow depending on the way a contradiction was defined as same cases of opposing basic states. Amazingly, not only the namings, but even the basic states are immaterial. In short, it doesn't make any difference what the lines contain, they might not contain anything at all, that is they can be regarded as single objects. But if they are such objects without elements, then how can some of them become contradictory? We don't care! We can ignore the internal features, and simply accept that some of them have a special relationship of contradiction. We don't have to dig deep, why some people hate each other, if all our goal is to make a survey about people's behavior. Just as sociology is limiting the psychological questions to proceed to statements about people's social relationships, we can do the same in math, by simply regarding relationships as sets.
Thus, in our matrixes, the lines will be regarded as singular objects among which the contradictions are not defined internally, rather will be given directly as a mere collection. The matrixes themselves are then also merely sets having their lines as elements. Choosing a line from each matrix is then choosing an element from each set. So we have three levels of elements. The biggest set is what was our set of matrixes, the elements of this are the matrixes and the elements of these are the old lines.



Picking one line, that is element, from each matrix, we obtained a choice set. We can even imagine that we actually pull out these choice elements and collect them as a choice outside:



Choice set

What we called contradictory choices, will be now also given as a collection, namely the collection of all so called "taboos". This sounds better than contradiction, because it reflects that they are to be avoided by mere convention or agreement. Each taboo is a choice, but unlike the above choice set, it doesn't have to be taken from all sets. In theory, of course, a taboo could be a total choice set too, but it's not typical. Indeed, the typical is that we have lots of little taboos, and then we have to avoid them all to get an aimed choice set. Of course, the smallest taboo would be just one element from a single set, but this is too small, all it would mean, to avoid that particular element. The collection of all taboos is the taboo set. Up until now, we just changed from matrixes to sets in general. And I already revealed that the lines were unimportant, that's why we have them as the lowest elements. So what was then special about the matrixes? Only two things:

1. Every matrix has only finite many lines.
2. A contradiction involves only two lines.

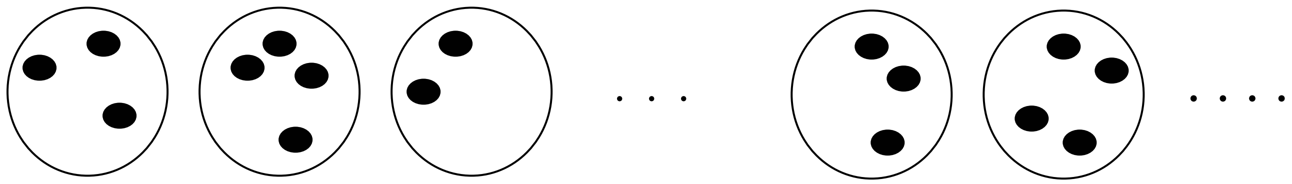
In our new concept:
1. Every choosable set has finite many elements.
2. All taboos have two elements.

Amazingly, that's all we need, in fact 2 can be used with "finite", instead of "two". Thus, our claim:

If an $\mathcal{S} = \{S_1 , S_2 , \ldots\}$ set of sets is such that:
1. Every S_α has only finite many elements, that is $S_\alpha = \{s_1^\alpha , \ldots , s_m^\alpha\}$.
2. We have a $\mathcal{T} = \{T_1 , T_2 , \ldots\}$ taboo set, that:
   every T_α has only finite many elements, that is $T_\alpha = \{t_1^\alpha , \ldots , t_m^\alpha\}$
3. From every finite $\{S_\alpha , S_\beta , \ldots , S_\gamma\}$ set, we can make a choice set $\{s_i^\alpha , s_j^\beta \ldots , s_k^\gamma\}$ that avoids all taboos.

Then, there is a total $\{s_1 , s_2 , \ldots\}$ choice from $\mathcal{S}$ that avoids all taboos of $\mathcal{T}$.

We already got used to the idea of imagining sets as a list. Indeed, the realities were also envisioned as such. But downwards, under each other to see the columns. Then, at the matrixes, instead of downwards, we imagined the list as left to right to see the continuing line choices. Here, again we should do this, but the s elements of S-s don't have to be imagined under each other anymore.



S_1 S_2 S_3 S_ω $S_{\omega+1}$

I continued the infinity of S-s with a new S_ω which is customary.
The $T_1 , T_2 , T_3 , \ldots , T_\omega , T_{\omega+1} , \ldots$ taboos look exactly the same as the S-s above, but don't forget that their elements are picking from some finite many S-s.
So, for example, if $T_3 = \{t_1^3 , \ldots , t_m^3\}$, then t_1^3 can be from S_5, t_2^3 from S_{21} and so on.
Avoiding T_3, then means that we can't pick this full collection! So we can pick t_1^3 from S_5 or t_2^3 from S_{21} and so on, but to pick all of these together is not allowed.
At the listing of realities, the fact that we didn't use the envisioned listing, was guaranteed by never using the envisioned order. Indeed, when we said bring such and such lines to the top, to make the "topping", we told exactly what kind of lines to choose. Here unfortunately, we'll use the order in case of the choosable $S_1 , S_2 , \ldots , S_\omega , \ldots$ sets, but not for the $T_1 , T_2 , \ldots , T_\omega , \ldots$ taboos.
This means a serious problem, for two reasons: First, we have to be careful what we use from this envisioned ordering from left to right. Secondly, we must prove that every set of choosable S sets could be put in such order. The first is quite easy to tell, because we'll only use one assumption about such list, namely that every beginning part of it has a next element.

$$\underbrace{S_1 , S_2 , \ldots , S_\omega , S_{\omega+1} , \ldots , S_{\omega\omega} , \ldots ,}_{\text{beginning}} \quad S_\alpha$$

The beginning can have a last element and then S_α is the next to it, or can be unfinished and then S_α is only next to the whole beginning.

The second question, that is how to list every set in this manner, is much more difficult and we'll deal with it later. This problem was only discovered after Set Theory and Logic started to find each other. Before, it was just assumed that if we pick elements from a set, then the pickings can be regarded as such order. After all, if up to a point, we didn't pick all elements, then we could pick new one, which is thus, the next after the beginning. Clearly, here we use some "timely" concept. And though it is plausible, it is not the same, as merely collecting sets by elements, which is more "space like". So the question was, can we eliminate this concept of time from Set Theory? The pickings were dependant on the beginnings that were "already" picked. So a perfect solution could be where we pick potentially next elements, right at the start, that is without time, and then let the sets themselves define their ordering. This sounds quite simple! For every subset of the set, lets choose an element outside the subset. This will be the potential continuing element if that particular subset is obtained as a beginning. Then, lets pick any s_1 element of the set as a start. $\{s_1\}$ has already a pre chosen outside element and that will be s_2. Then, $\{s_1 , s_2\}$ again has a pre chosen continuation s_3. And so on, $\{ s_1 , s_2 , . . . \}$ again is a subset, unless it's the full set, so it has again a pre chosen continuer, s_ω. And so on, as we see, every beginning will automatically determine the next element. Lets observe that not every subset will appear as beginning, so not every potential continuation will be used. The s_1 starting element will initiate and determine the full process of continuations. So now, the time process is automatic. But this is not the end of the story. Time is still hidden, in our proof that such self selection exists. In order to eliminate it completely, we should be able to obtain this final "time sequence" as a mere "spatial" set, obtained from the initial choice set for the subsets and s_1. This can be done, and that was the first big success of axiomatic Set Theory. At the same time, it turned out that even this initial choice, that is, simultaneously picking elements from sets, was something that had to be stated as an axiom. That was the point where everybody went crazy, forgot about the whole positive result of time elimination and just like parrots, repeated the new axiom.

But lets return to our $S_1 , S_2 , . . . , S_\omega , . . .$ list and using it, tell the $s_1 , s_2 , . . . , s_\omega , . . .$ elements that can be picked from them, avoiding all taboos. Already, the first choice, s_1 from S_1, shows the problem. After all, if S_1 has an element that is not element of any taboo, then it's fairly obvious to choose that. But, it very well may be, that all elements of S_1 are in taboos and yet, we can avoid taboo combinations. But how can that boil down to an actual perfect choice from S_1.

The solution is so heuristic and simple, that it moves the heart of any truth loving potential mathematician.

Our third claim was, that for any finite many S sets, there is a taboo avoiding choice. So then, lets pick an element of S_1, say s_1^1, and check whether using this s_1^1 as a fix element, our third claim, still remains true. In other words, for any finite many S-s, we can pick a second, third and so on, elements from them to add to the fix s_1^1, so that no taboo is appearing. Of course, we might be unlucky, which means that for s_1^1, there are some $\{S_{\alpha_1} , S_{\beta_1} , . . . , S_{\gamma_1}\}$, so that no matter which we pick from these, the $\{s_1^1 , s_j^{\alpha_1} , . . . , s_k^{\gamma_1}\}$ is always containing taboo. In this case, we choose an other element of S_1, say s_2^1. And so on, we'll try until we succeed. Amazingly, we must succeed!

Indeed, if all elements of S_1 were unsuccessful as fix element, then that would mean that for each s_i^1, there were a $\{S_{\alpha_i} , S_{\beta_i} , . . . , S_{\gamma_i}\}$ set, giving taboo unavoidable continuations. But then, lets combine all these sets into the $\{S_\alpha , S_\beta , . . . , S_\delta\}$ total of them. It is still a finite set and then adding S_1 to it, that is $\{S_1 , S_\alpha , S_\beta , . . . , S_\delta\}$ is one that can't avoid taboo with any pickings. This then, would contradict our third assumption. So, there had to be a lucky s_i^1, which is thus our first pick s_1. Now this s_1 is fix, so from S_2 we can try again all s_i^2, so that s_1 , s_i^2 can be continued with any finite picks. And of course, again we can see that such s_i^2 must be! Indeed, otherwise now not the original third assumption, but rather our previous result that s_1 is continuable, would be contradicted. Thus, step by step we create $s_1 , s_2 , . . . ,$ that is a first infinite sequence is chosen.

Before, we continue to choose s_ω, it's interesting to pin point why we needed our three assumptions. The third was obvious and was the base of our whole construction of choices. The first was used to be able to combine the finite sets for the unsuccessful s elements into a still finite total. This then created a contradiction if applied to all s elements of an S. This is a pretty indirect usage and therefore it would be educational to show that having infinite S-s could ruin our claim. I show now that having just one infinite S_1, can already ruin it. Indeed, let $S_1 = \{1 , 2 , 3 , . . . \}$ and $S_2 , S_3 , . . .$ whatever we want them. Let the taboos be: $\{ 1 \}$, $\{2 , \_ \}$, $\{3 , \_ , \_ \}$, $\{4 , \_ , \_ , \_ \}$, ...
Here, _ stands for any elements picked from $S_2 , S_3 , . . .$ In short then, the taboos are finite sets that have as many elements as the picking from S_1. The third condition is true, because if S_1 is not picked, it's obviously not a taboo, and if S_1 is picked, then simply pick a different number than the number of S-s used. And yet, no matter what we pick from all the $S_1 , S_2 , S_3 , . . .$ it will start with an m picked from S_1 and thus, $\underbrace{\{m , \_ , \_ , \_ , . . . , \_ \}}_{m}$ will be appearing in it.

Finally, we should see where the second condition was used. It's quite hidden, definitely not in the pickings of $s_1 , s_2 , . . .$, rather in the final claim that these together are good. Indeed, they are only good because if they would contradict a taboo, then that taboo being finite, would be contradicted already up to an s_n. To create a counter example, for allowing even just one infinite taboo, let that taboo be $\{1 , 2 , 3 , . . . \}$ and let the S sets be $\{0 , 1 \}$, $\{0 , 2 \}$, $\{0 , 3 \}$, . . .
The single finite taboo should be $\{ 0 \}$. Indeed, then for any finite collection of S-s, we can avoid 0 by choosing the natural numbers from them. But for the total $S_1 , S_2 , . . .$ either we choose a 0, then contradicting the finite taboo or we pick the naturals and then contradict the infinite taboo.
Now we can return to continue our proof for $S_\omega , S_{\omega+1} , . . .$ Seemingly, we have a problem! Before, we used the earlier picks as a fixed set to be continued for longer, finite picks. But now $s_1 , s_2 , . . .$ is infinite, so adding s_1^ω to them is infinite too. The solution is exactly what we used in our last examination of the second condition.
Now the finiteness of taboos will be explicitly used already for the new pickings. Indeed, even though $s_1 , s_2 , . . .$ is infinite, if these plus a picked s_1^ω from S_ω plus some finite $\{S_\alpha , S_\beta , . . . , S_\gamma\}$ later sets with all their choices are containing taboos, then all these taboos are involving only finite many elements from $s_1 , s_2 , . . .$ So combining all these for all the encountered taboos are still finite too. Thus, we have a finite set containing elements from $s_1 , s_2 , . . .$ plus s_1^ω plus all elements from later $S_\alpha , S_\beta , . . . , S_\gamma$. Now if we assume the same for all elements of S_ω, then combining all these finite sets we still end up with a finite set, having some elements from $s_1 , s_2 , . . .$ but now having all elements from $\{S_\omega , S_\alpha , S_\beta , . . . , S_\delta\}$.
This means that those elements from $s_1 , s_2 , . . .$ are containing taboos with every choice of $S_\omega , S_\alpha , S_\beta , . . . , S_\delta$. But this is contradicting how we always picked the elements, namely with the last s_n element from $s_1 , s_2 , . . .$
So, we can repeat our heuristic idea of picking an element from S_ω say $s_i^\omega = s_\omega$, so that $s_1 , s_2 , . . . , s_\omega$ can again be continued with pickings from any later $S_\alpha , . . . , S_\gamma$ without taboo.
Of course, after $s_1 , s_2 , . . . , s_\omega , s_{\omega+1} , . . .$ a new element comes again, namely $s_{2\omega}$, then $s_{2\omega+1}$ and so on. The notation of new indexes is immaterial. The point is that after every beginning, there is a next new element. That is, until we finish all S-s, and thus, obtain a full choice.
Lets emphasize again, the heuristic idea:
Instead of picking new elements, that avoid taboos with the already picked elements, we pick new ones that avoid taboo with all the picked ones plus all finite many potentially new ones.
This theorem at once means that we can also pick lines from all matrixes without having contradictory basic cases.

9. Quantified Matrixes

Our fundamental theorem above dealt with the reverse realization of a matrix. Earlier we dealt with quantifications of reality. So we have three things, quantification, reality and matrix. They can even be visualized in this order. Namely, quantification on top of a list, and then a matrix or its realization by the lines of the list. For this, we had to identify the variables of the matrix with the columns, so this lettering should be on the top, even above the quantors.

$$\begin{array}{l} \underline{(x\,,y\,,z\,,\,.\,.\,.\,)} \\ \forall \;\; \exists \;\; \exists \;\; .\,.\,. \\ (1\,,3\,,\pi\,,\,.\,.\,.\,) \\ (0\,,1\,,2\,,\,.\,.\,.\,) \\ . \\ . \end{array} \qquad \left[\qquad\right]\left[\qquad\right] .\,.\,.\,.\,. \left[\qquad\right] .\,.\,.\,.$$

This picture begs the question, whether we can skip the middle reality and use a quantification directly for a matrix:

$$\forall\, x \;\; \exists\, y \; .\,.\,. \left[\qquad\qquad\right]$$

So here we are, back to our full language of logic, but now the meaning is obvious too.
Such statement is a possibility, if there is a possible insertion of reality between the quantification and the matrix. Or in negative, a statement is impossible if there is no reality that satisfies the quantification and reverse realizable for the matrix. We made a little hidden simplification. The quantification of reality could have been a non widening dependence with arrows, but our new notation jumped right into the simple consecutive or widening quantifications only. The reason for this is, how we struggled with our proofs in the minimal reality in 5. Lets remember, that for a given reality, it was even easier to use the non widening dependences, because they inherited automatically, but when we wanted to create a reality, we had to go in widening or consecutive order.
Now we want to eliminate the middle reality, so that's why we at once assumed simple quantification in order of appearance. I even gave a line by line build up of quantifications that contained the required reality. The two steps to create the lines were $\exists$ and $\forall$ namings. The end result was that every quantification had a reality, so this would seem to make our new quantification of matrixes fairly pointless, because the possibility or impossibility of such would merely depend on the reverse realization of the matrix. So, in short, the quantification is unimportant, and only the matrixization is the problem. But even that isn't a big problem, because we had our taboo avoidance theorem that reduced it to the finite subsets of the reality. We made a little oversimplification here though. It's true that for every quantification we can regard our heuristic step by step reality and then for the possibility of its matrixization, the finite subsets of that reality. We can even combine it by saying that if the $\exists$ and $\forall$ namings of the quantification don't lead to finite many lines, that are not matrixizable, then there is a full matrixization. However, the reverse is a problem now. With one fixed reality the reverse was obvious. If a finite subset was contradictory, then the full was such too. But if only a quantification is required, then just because our step by step reality leads to a finite contradictory path, it doesn't mean that the quantification is impossible with the matrix. Indeed, maybe some other realities for the quantors would not be contradictory. Thus, exactly happened what I fortold, that logic is progressing toward the question of impossibilities. Before, the impossibility was easy and the big result was for the possibility. Now, the impossibility is the challenge. Unless of course, our created reality by the $\exists$ and $\forall$ namings is such that it is a perfect representative of all possible realities, as far as matrixization is concerned. In other words, if our reality is contradictory in some finite cases, then all others would be contradictory too. Amazingly, that's the truth. And the reason for it, is quite simple. The $\exists$ namings created new names. Any other reality would be merely a simplification, by having some of our new names to be an earlier one.

Now in the matrixes, looking at such name reductions, it is obvious that if two basic cases were contradictory, then reducing the names, they still remain contradictory. The opposite is obviously not true, that is combining names can bring in new contradictions. If it wasn't so, then all we had to do is combine all names, that is write one single name into all variable of our matrix and then check if it is contradictory or not. And of course, a contradiction in such "single-named" matrix, would only be if every line contained opposite basic states $A(\ldots)$, ... , $\neg A(\ldots)$, regardless of the variables in the brackets, because they become the same by the single naming. Or in reverse, any matrix having a line without opposite states would be a possibility. Now here we again made an error of reversal, because this reverse is actually true. Indeed, if a matrix has such line without opposites, then from any set of namings we can simply choose this line from all of the matrixes and obviously no opposite case can appear, since there are no opposing basic states already. This of course, then makes any quantification of such matrix possible too, because no naming brings contradiction.
Amazingly, even the non true reverse, that is the impossibility of uni-named contradictory matrix is true for special quantification, namely the strongest one, that is claiming $\forall$ for all variables. This is so because in realities that satisfy such quantifications, all combinations of names must appear and this includes lines with all the same name. And of course, if such is contradictory in the matrix, then the full list is so too automatically. We might say that's not so surprising, after all a full $\forall$ quantification is something extremely strong, and thus, easy to refute. But this is not quite true! A simple version of number theory can be formulated with all such universal statements as axioms. This simple number theory became famous after Godel proved that within that, we can't prove that it is non contradictory. Now if we combine those axioms into one statement, it is still universal and has a matrix that can be easily checked whether it has a line without opposite states. Thus, Gödel's result couldn't be true.

Our error was that we forgot about two things, we introduced in the language of logic, and both of which appear in number theory. One is the usage of functions, or as we call them in number theory, the operations $+$ and $\times$ and the other, the $=$ relation. Why is this $=$ different from other basic states like say $<$, which is also used in number theory? Equality is different because it can become contradictory in itself! Indeed, if $x = y$ appears in a matrix, and we write two different names in x and y, then $n_1 = n_2$ is false in itself. Thus, this changes everything about the occurring opposite pairs for contradictions.

<u>10. Matrixes With Functions and Equality</u>

They say that the essence is in the details, so we shouldn't regard these two as nuiscanes, rather as the sugar and salt that make the matrixes really powerful. In fact, it is true, throughout in math that functions can make things easier, thus sweeter, while equality, harder.

To be continued . . .