Nim – The Game of Take

D In the followings, a “set” will mean a few lines of objects, like cards or match sticks.

If you use paper and pen instead of objects, then a column of single numbers can replace the set:

□ □ □ □ □ □	6
□ □ □	3
□ □ □ □ □ □ □	7
□ □ □ □ □ □ □ □ □	9

The crucial action of the game is the “take”, from any line, that is, a subtraction from any number.
Only one line can be altered, but one can take the whole line, if he wishes.
In the paper version, this means subtraction, to get a 0.
The taken objects are put aside, regardless who takes them, because they are not the goal of the game. Instead, the aim is to take the last objects, that is leave nothing on the table. On paper, this means to achieve all 0-s.
A game history with numbers can be:

6		6	⇒	0		0		0		0		0		0		0
3		3		3		3		3	→	1		1	→	0		0
7		7		7		7	⇒	3		3	⇒	1		1	⇒	0
9	→	1		1	→	0		0		0		0		0		0

So ⇒ wins.

D “Groups” are the: 1 , 2 , 4 , 8 , 16 , . . . numbers or that many objects.

It’s a simple fact that any number can be expressed from groups uniquely. Easiest to see this by making it from decreasing groups. The first group is the biggest that is not larger than the number. Subtracting this, the left over must be smaller and a new largest group can be subtracted again and so on. With the object set, we could use imaginary dividing lines, while with numbers, turn them into additions:

□ □ □ □ □ □	□ □ □ □ | □ □	6 = 4 + 2
□ □ □	□ □ | □	3 = 2 + 1
□ □ □ □ □ □ □	□ □ □ □ | □ □ | □	7 = 4 + 2 + 1
□ □ □ □ □ □ □ □ □	□ □ □ □ □ □ □ □ | □	9 = 8 + 1

D A group is even or odd in a set , according to the number of appearances in the set.

In our set above, the group 4 is even, while all the others, 8 , 2 , 1 are odd.
A set is even if all groups are even in it.
A set is odd, if it’s not even, that is some groups are odd in it.

T 1.) There is only one unique line that can be added to an odd set to make it even.

2.) An even set always turns odd if a whole line is taken from it.
3.) An even set always turns odd if any line is altered in it.
4.) An even set always turns odd by any take.

P

1.) Let the odd groups in the set be g_1 , g_2 , ... , g_m .
These and only these must be altered in their occurrence parity. A new line can only add single occurrences. So the new line must be $g_1 + g_2 + . . . + g_m$.
2.) Let the line be $g_1 + g_2 + . . . + g_m$. Removing the line, these groups turn odd.
3.) Removing the line, the set turns odd by 2.).
Then by 1.), only the original line could make it even. So any other makes it odd.
4.) Special case of 3.).

T

From any odd set, we can always take to turn the set into even.

P

Suppose that the odd groups in the set are in decreasing order: g_1 , g_2 , ... , g_m .
These odd groups may appear in different lines, so taking one out from each in a single line is hopeless. Luckily, taking one out is not the only way we can alter the oddness to evenness. Namely we can add one. Of course the total must be a take.
This will always happen if we make sure that the largest g_1 odd group is taken.
So all we have to do is find any line where g_1 appears. Then take this and also all the appearing a_1 , . . . , a_k odd groups from it. But this won't be our final take, because the non appearing n_1 , . . . , n_j odd groups must be added, that is the take reduced by these. So our final take is: $g_1 + a_1 + . . . + a_k - n_1 - . . . . - n_j$.
For example, in our above set, the odd groups are, $g_1 = 8$, $g_2 = 2$, $g_3 = 1$.
$g_1 = 8$ appears only in the last line, with the $a_1 = g_3 = 1$ while $g_2 = 2$ is the non appearing n_1. So, our take is $g_1 + a_1 - n_1 = 8 + 1 - 2 = 7$

$$\left.\begin{array}{lll} 6 & 6 = & 4 + 2 \\ 3 & 3 = & 2 + 1 \\ 7 & 7 = & 4 + 2 + 1 \\ 9 \rightarrow & 2 = & 2 \end{array}\right\} \text{ even}$$

D

These two theorems provide a strategy for winning the game of take.
Indeed, all we have to do is always take by our second theorem, to turn the set even.
Then no matter what our opponent takes, will create an odd set by first theorem's 4.).
The final goal of the empty table is of course an even set, so our opponent can not obtain it. Since the number of objects decreases, we must reach the last take.
The only problem is to get into one odd set before our turn is to take.
The following is a "sneaky" method for this:
We ask our potential opponent to place objects in lines as he wishes.
While we explain the rules, we group the lines mentally and establish whether the set is even or odd. A set is much more likely to be odd than even, so usually we start by showing a seemingly arbitrary example of taking, which actually turns it even.
If accidentally our opponent's layout is even, then we ask him to take first as a further sign of arbitrariness. This of course will make the set odd and we can turn it even.
After many games, the question of who starts will be raised and altered but even if we let him start from an odd set, usually he won't get into an even, so we can still win.
We can confuse our audience even more with a new variant of the game where the last taker is not the winner rather the loser. We can win that game too.
Amazingly, the strategy remains the same almost to the very end. Namely, we play the same way until an odd set appears with all lines being 1 except one n.
The rule of evenness would dictate to take out the whole n line or leave just one element in it to create even many 1 lines. But here we do oppositely, creating odd many such lines. This will force our opponent to take the last single object.

R The obvious question is whether our strategy is the only one. Of course, we didn't define what a strategy means in general. But this is the easy part, and then we expect that our strategy using the groups can be shown to be the necessary one.

Instead, we'll show that there is one single strategy. This of course, proves that the group strategy is the only one. The strange thing is that that we don't have to mention groups at all, to prove the existence and uniqueness. Of course a much easier proof of the existence is by using the group strategy. But then it is pulled out of the hat.

So, here we encounter an example of a wider problem in classical mathematics, the ad-hoc existence proofs. We claim that something exists and we simply give an example for it. This in itself is not strange. Something can exist with a multitude of cases, so one example must be ad-hoc. The strangeness comes in if the existence is unique or it leads to some other uniqueness.

In our present subject, the concrete group strategy was a natural start, because we had to show how to win. But if we started from strategies in general, then proving the uniqueness still doesn't give the existence. Then, the concrete group strategy is the "easy" ad-hoc existence proof, that is unique too. So the strangeness is, why uniqueness doesn't imply the actual strategy already. We have to guess it externally.

In other cases, the ad-hoc existences are not unique, so quite acceptably, they are ad-hoc, but they become part of proving something else, that is unique. So the existence is all that matters, but we can't avoid the ad-hoc examples for them.

A famous example is the splitting of the primes.

All primes, except 2, are odd, and so a natural splitting of these by a new "parity" can mean the relation to the 4-multiples. That is, being $4k-1$ or $4k+1$:

$$3 \quad \textcircled{5} \quad 7 \quad 11 \quad \textcircled{13} \quad \textcircled{17} \quad 19 \quad 23 \quad \textcircled{29} \quad 31 \quad \textcircled{37} \quad \ldots$$

I circled the $4k+1$ ones. The amazing fact is that all these, can be written as :

a^2+b^2 square sums: $5=1^2+2^2$, $13=2^2+3^2$, $17=1^2+4^2$, . . .

What's more, these square sum forms are unique for all these circled ones and the non circled $4k-1$ primes have no such forms at all. So clearly, we have a deep rule that must have some internal cause. When Gauss realized the complex numbers as plane numbers, then he used this vision for proving the fundamental theorem of algebra, though he never explained his new vision. Probably, because he regarded the new numbers as physical, but couldn't derive any physical meanings yet. Today, they are in physics everywhere, so his intuition was correct. As a different direction of application, he also introduced the complex or plane whole numbers. They are simply the grids of a Descartes system. Addition and multiplication can be defined and a new primeness appears among grids. Surprisingly, only some of the old primes of the x axis remain primes as grids. A natural prime grid on the x axis can only be product of symmetrical (a, b) , $(a, -b)$ grids with the product value of a^2+b^2.

So, exactly the square sum primes become composite grids. This includes the simplest $2=(1, 1)(1, -1) = 1^2+1^2$ and the simplest odd $5=(1, 2)(1, -2) = 1^2+2^2$.

This gives a new meaning for the square sum primes, but still doesn't provide a proof for the $4k+1$ coincidence. So in a sense, the grand vision of complex primes, makes it even more ad-hoc, how the proof will go. The prime factorization among grids does explain that simple a^2+b^2 square sums, that is with a, b being relative primes, can only have same kind of prime factors. In fact, the grand picture also explains that these multiply into same, so all factors are simple square sums.

The prime factors inheriting square sumness, means that the $4k+1$ coincidence would follow if all $4k+1$ primes must divide some special simple square sums.

A choice of this "special" can be $n^2+1=n^2+1^2$ because 1 is relative prime to all numbers. And indeed, all $4k+1$ primes do divide some n^2+1 numbers.

Unfortunately all the proofs for this, go by finding some ad-hoc $n^2 + 1$ constructions from $4k + 1$, through some other theorems.
The reason for these didactically objectionable proofs is that we only regard fragmented worlds as natural realities. For example, the complex whole numbers as grids are still not the widest reality for investigating primalities.
But here, with the “game of take”, we don’t know what wider reality lurks behind.
The question whether the widest pictures can always be reached without becoming so abstract that we lose the pictures themselves, is the real question.
I do believe that the didactical aspect is fundamental, but we are not possessing a framework to approach it. So though now, everything in our world is becoming more and more abstract and arbitrary, there will be a turnaround to simplicity and perfection. But there is an even darker side to the present:
The arbitrariness of mathematical arguments is the real cause of the difficulty for the common sense. In spite of math basically being common sense itself. The seemingly more math oriented kids are simply ones who can accept the arbitrariness and can jump to abstractions, where the common sense meanings finally kick in. But many deeper minds turn away from math at the start. The demagogue “wisdom” of “talent always finding its way” is false, because the education system is fundamentally faulty.
There is a more mystical element of information from the other side. Obtaining visions without education at all. But even these abilities to become mediums, require a preparation through common sense, cause and effect. So actually, a war is going on:
Society intentionally tries to block the truth, so individuals have to break through.
This subject matter, the “game of take” is a perfect example. If you want to see this, just look at what over-complicated abstract junk you will find on the internet.
But now we have to return to our limited reality, and ad-hoc proofs.
At least I explained how and why we proceed as we do.

D

A collection of sets is a full strategy, if :

a.) Every set outside the collection can be moved into the collection by taking from it.

b.) The empty set is in the collection.

c.) Every non empty set in the collection, moves out by taking from it.

A collection of sets is a strategy, if :

b.) , c.) stands as above but instead of a.) :

d.) Every set moved out of the collection by c.) can be moved back by a take.

So a strategy guarantees the win, only if we somehow get into it.
Every full strategy is a strategy and every strategy is a full one if a.) holds.

The simplest strategy is the emptiness plus two lines with one-one object, or with numbers:

$$E_0 = \left\{ \begin{matrix} 0 & 1 \\ & 1 \end{matrix} \right\}$$

A less obvious is :

$$\left\{ \begin{matrix} 0 & 1 & 2 & 3 & \\ & 1 & 2 & 3 & \end{matrix} \; \ldots\ldots \right\}$$

Indeed, whatever our opponent takes from one of the lines, we take from the other.
This will reduce to the previous E_0 .

T

1.) If S is a set in an E_1 strategy, then taking from S, not only can't stay in E_1, but it can't move into any other E_2 strategy either.

2.) The $E_1 \cup E_2$ combined strategies from two, is also a strategy.

3.) Every E strategy is subset in any F full strategy.

4.) There can be only one F full strategy.

P

1.) Suppose that taking from S in E_1 we get S' in E_2.
Then by d.), from S' we can take to get an S'' back into E_1.
This S'' would also have to be out of E_2 by c.).
So, S'' is a new smaller set in E_1, that can be moved into E_2 by d.).
Repeating this we would have infinite many, smaller and smaller
S , S'' , S'''' , . . . sets in E_1. An other way of saying this indirect argument is that such S set would be a start to win against each other by both strategies.

2.) By 1.), all elements of $E_1 \cup E_2$ will go out after taking.
The return, that is d.) and the empty set b.) are trivial for $E_1 \cup E_2$, from being true for E_1 , E_2 separately. So b.) , c.) , d.) all stand for $E_1 \cup E_2$.

3.) Let S be an arbitrary element of E . By 1.) S can not go into F.
But, by a.), if S is out of F, it can go into F. So S must be in F.

4.) For any two F_1 , F_2, each has to be part of the other by 3.).
So in fact, these two are the same.

T

If an E strategy is not full then there is S set not in E that can be added to E to get a new wider strategy.

P

Lets regard all those sets that can not be moved into E by taking.
This collection is not empty by E being not full. There has to be elements in this collection with minimal objects. Any such will give a desired S.
Indeed, they have to move into sets outside E that are movable into E.
At the simplest strategy, $E_0 = \left\{ \begin{matrix} 0 & 1 \\ & 1 \end{matrix} \right\}$ it is easy to verify that:

All sets with three or less objects are either in E_0 or can go into E_0 by a take.
But the following two sets with four objects can not move into E_0

2	1
2	1
	1
	1

So adding either of these to E_0 we get a new strategy. By 2.) we can add both too.

T

There is a unique full strategy.

P

The union of all strategies is a strategy by 2.) and must be a full by the last theorem.
This is unique by 4.).
An alternative proof for the existence is our group strategy introduced earlier.
The existence and uniqueness proofs did not imply the group strategy directly!!!