Power Sum Formulas

Definition

By the power sums we mean the same p powers of the natural numbers added.

That is: $1^p + 2^p + 3^p + \ldots + n^p = \sum_{1}^{n} k^p = \Sigma k^p = S_p$

The simplest natural power is $p = 1$ but in the "modern" approach the number 0 is regarded also as a natural number and this power is defined to have the value 1 for all bases.
So we have actually two simplest cases as:

$1^0 + 2^0 + 3^0 + \ldots + n^0 = 1 + 1 + \ldots + 1 = n$

$1^1 + 2^1 + 3^1 + \ldots + n^1 = 1 + 2 + 3 + \ldots + n = ?$

Of course, by regarding 0 as natural number we should have included it as base too and so the simplest power sum should be:

$0^0 + 1^0 + 2^0 + \ldots + n^0 = n + 1$

For other than 0 powers a 0 base has always 0 value and so in those power sums this first member is irrelevant. This shows that not everything is so logical in the abstract agreements.
And yet the abstractions can help to solve the natural questions too.
In fact, this article is a perfect exemplification of this.

The child Gauss solution

Back to business, we should calculate the second simplest sum of the first n naturals.
That's what the elementary school teacher of Gauss gave for the smaller kids to calculate up to hundred as task so she can teach the older ones.
But Gauss came up with the answer in few minutes.
He imagined the naturals backwards too from hundred to 1 underneath the increasing ones:

1	+ 2	+ 3	+ 4	+ . . . +	100 = ?
100	99	98	97		1

As we can see the sums of the two numbers under each other are always 101.
And so the twice of the total is actually 100 x 101 = 10100. And so ? = 5050.

Using Gauss' trick in a modern form should go like this:

$$2S_1 = 2\Sigma k = [\,1 + 2 + \ldots + n\,] + [\,n + (n-1) + (n-2) + \ldots + 1\,] =$$

$$[\,1 + n\,] + [\,2 + (n-1)] + [\,3 + (n-2)] + \ldots + [\,n + 1] =$$

$$[\,n + 1\,] + [\,n + 1\,] + \ldots + [\,n + 1\,] = n\,(n+1) = n^2 + n.$$

And so $S_1 = \frac{n^2}{2} + \frac{n}{2}$.

The squares sum formula through the sum of odds, first part, Galileo and Kepler

We turn to S_2 as a detour because soon we find a general solution to S_p.
The reason to show this detour is not merely to see how surprising some tricky ideas can be but also that this particular trick had a physical relevance too.

The trick as the title reveals it, is the summing of merely the odd numbers.
Why not the evens? Well, we'll sum the evens too as a sub trick inside the main one.
But the odds are more important not just because surprisingly their formula is simpler but also because they have the mentioned physical significance.
Galileo by his experiments at the Pisa tower observed that the falling bodies in the consecutive equal time intervals have increasing falling distances exactly as the increasing odd numbers.
So if the fall in the first τ time interval is δ then in the second third and so on τ time intervals the falls will be 3δ , 5δ , . . . and so on.
Unfortunately, Galileo did not realize that the fact of the odds combined giving the squares: 1+3= 4 , 1+3+5= 9 , and so on, can lead to an important continuation of his observation.
Namely, that the d total falling distance is proportional to the square of the t falling time.
So $d = c\,t^2$ and thus we can calculate this d for any t times not just the whole multiples of a chosen τ unit. If for example τ is chosen as a second then at the $t = 3.75$ second the total fall is $d = c\ 3.75^2$. Of course, we also have to show how this c constant relates to δ that Galileo needed to calculate the falling distances. Observe that:

$$d = \delta + 3\delta + 5\delta + \ .\ .\ + (2n-1)\delta = \delta\,[\,1 + 3 + 5 + .\ . + (2n-1)] = \delta n^2 = \delta\left(\frac{t}{\tau}\right)^2 = \frac{\delta}{\tau^2}t^2 .$$

So this $\frac{\delta}{\tau^2}$ is that should approach our mentioned c constant if we chose small enough τ.

But why? Well, the initial step could be to realize that $\frac{\delta}{\tau}$ must approach 0.
Indeed, this ratio is the speed at the start and we let our body fall by merely letting it go and not giving it any initial push down. So we entered the concept of momentary speeds and then we can also ask how this obviously increasing momentary speed changes. Then we can realize that it must grow linearly, that is proportionally with time. This also means that the change of these momentary speeds under fix small time intervals, that is the speed of the speed change is a fix constant. Then we can realize that this, what we usually call as acceleration is the only part in a motion that requires a force. So the mere fix speed is a natural motion without force.
And finally, we still must realize how a $v(t)$ momentary speed function or velocity can be calculated from a given $d(t)$ distance function. That is, invent derivation that shows why a square proportional $d(t) = c\,t^2$ gives a $v(t) = d'(t) = 2c\,t$ velocity change which then indeed gives an $a(t) = v'(t) = 2c$ fix acceleration. So if the fix force of gravity on our Earth is forcing a fix $g = 9.81$ acceleration with using meters and seconds then c must be $\frac{g}{2}$ and the falling law becomes: $d = \frac{g}{2}t^2$. And then it all makes sense! Indeed, g is how much the speed increases in a second. In the first second it starts from 0 speed and so the average speed in this first second is $\frac{g}{2}$ and so at the end of the first second the fallen distance is $\frac{g}{2}$ = about 5 m.
In the second second the speed will increase from g to $2g$ and so the average speed is now $1.5g = 15$ meter per second and this causes 15 meter drop in this second.
Then we get 25 meter by similar argument and so the odd consecutive falling distances are observed as 5 , 15 , 25 , 35 , . .
Only Newton was able to make these steps from using Galileo's and Kepler's observations.
And in the case of Kepler, the necessary new steps were even more drastic because the orbiting motions had to involve vectors as velocity and acceleration.
But the underlying connection to the forces were again the essential plausibility ground.

It is amazing and not well known at all how close Kepler was to discover gravity and dynamics. He said that if we could stop the moon in its motion around the Earth it would start to move towards the Earth just as we towards the moon and meet at our mass proportional "middle".
But now comes the most important part:
This error of regarding the orbiting as a natural base that somehow changes into this attraction is a sign of the common fundamental obstacles in everybody's thinking.
The alternative correct logic that attraction can cause the orbiting, needs the crucial abstract step that has to be walked again and again using the vectors to become our new plausibility.
But this does then become the new plausibility indeed. So correct use of the abstractions is vital.
My favorite pastime was to talk to high school physics teachers just finishing uni and ask why the moon doesn't fall to the Earth. A usual answer was that the centripetal and centrifugal force balance each other. Telling that by this logic then the moon should be travelling straight with a fix speed by Newton's first law, made them realize that they understand nothing.
Nature is working exactly in accordance with our human thinking but not in a primitive sense that the non thinking social system regards as teaching.
Neil deGrasse Tyson is the perfect example of the destructive phony "educators".
Believes that nature doesn't care what we find plausible, used the analogy for $E = mc^2$ the two sides of a coin and I could list more but the essence is that in his "master class" ad he promises to teach the potential customers "how to think". Scumbags rule this world!

The squares sum formula through the sum of odds, second part

First of all we must prove what we already used above, that the odd sums are indeed the squares.
We start from scratch so ask: $1 + 3 + 5 + \,.\,.\, + (2n-1) = \Sigma(2k-1) = ?$
As we mentioned, our sub trick will be using the evens:

$$1 + 3 + 5 + \,.\,.\, + (2n-1) = (2-1) + (4-1) + (6-1) + \,.\,.\, + (2n-1) =$$

$$(2 + 4 + \,.\,.\, + 2n) - (1 + 1 + \,.\,.\, + 1) = 2(1 + 2 + \,.\,.\, + n) - n = n^2 + n - n = n^2.$$

The main goal to sum the squares, will again use two tricks.
One by the just proved fact is that:

$$S_2 = 1^2 + 2^2 + 3^2 + \,.\,.\, + n^2 = 1 + (1+3) + (1+3+5) + \,.\,.\, + (1+3+5+\,.\,.\,+(2n-1))$$

$$= n \times 1 + (n-1) \times 3 + (n-2) \times 5 + \,.\,.\, + 1 \times (2n-1) = ?$$

Feels like a dead end because seems harder than the original problem of S_2.
But now comes the second trick, adding to this twice the squares backwards. And so: $3S_2 =$

$$\begin{array}{ccccccccc} n \times 1 & + & (n-1) \times 3 & + & (n-2) \times 5 & + \,.\,.\, + & 1 \times (2n-1) & + & \\ 2n^2 & + & 2(n-1)^2 & + & 2(n-2)^2 & + \,.\,.\, + & 2 \times 1^2 & & = \end{array}$$

$$n[1+2n] + (n-1)[3+2(n-1)] + (n-2)[5+2(n-2)] + \,.\,.\, + 1 \times [(2n-1)+2\times 1] =$$

$$n[2n+1] + (n-1)[2n+1] + (n-2)[2n+1] + \,.\,.\, + 1\times[2n+1] =$$

$$(2n+1)[n + (n-1) + (n-2) + \,.\,.\, + 1] = (2n+1)S_1 = (2n+1)\left(\frac{n^2}{2} + \frac{n}{2}\right) = n^3 + \frac{3n^2}{2} + \frac{n}{2}.$$

And so $S_2 = \dfrac{n^3}{3} + \dfrac{n^2}{2} + \dfrac{n}{6}.$

An amazing application of the squares sum formula, "Letting the bag fall"

Two people hold a bag of cement weighing 40 kp by placing their hands under the two ends. One pulls out his hand. How much w weight the other person must hold at this moment?

The other person's hand can be replaced by a scale so this force is very well measurable.

Interestingly, the intuitive answers can be: full 40 kp, same 20 as before, finally less.

This last is the correct answer but how much less is the detail we'll calculate too.

We'll be more general, so regard any d long and m mass object but regard its weight also as m so measure it in kp.

At the drop of one end it starts to rotate around the other kept end.

The $F = ma$ law of acceleration for a rotating body becomes $T = M\alpha$ where α is the angular acceleration, $T = \Sigma F r$ = torque and $M = \Sigma m r^2$ = rotational inertia.

To calculate T we can imagine the m weight force at the center and apply $\frac{d}{2}$ as r.

So $T = \Sigma F r = m\frac{d}{2}$. But $M \neq m d^2$ or $m\left(\frac{d}{2}\right)^2$ so the inertia can not be calculated by replacement similarly, rather must be calculated gradually.

So we split the rod into n many small pieces and regard Σ indeed as $\sum_1^n$.

Thus the M_e = inertia around the end as axis =

$$\Sigma \frac{m}{n}(k\frac{d}{n})^2 = m\frac{d^2}{n^3}\Sigma k^2 = m\frac{d^2}{n^3}(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}) = m\frac{d^2}{3} + md^2(\frac{1}{6n} + \frac{1}{2n^2}) \sim m\frac{d^2}{3}.$$

Regarding arbitrary big n this final approximation becomes the exact value.

To solve our problem we must regard as a new second axis the center of the rod too.

From here the equal counter force of our questioned w will appear as a turning force upward. But for the rotational inertia around the c center the $\frac{m}{2}$ mass and $\frac{d}{2}$ radius must be used for both sides applied according to our previous result.

Thus the M_c = inertia around the center as axis $= 2\frac{m}{2}\frac{\left(\frac{d}{2}\right)^2}{3} = m\frac{d^2}{12}$.

Luckily, the angular acceleration is the same for both axis. And so:

End axis equation: $m\frac{d}{2} = m\frac{d^2}{3}\alpha$. Center axis equation: $w\frac{d}{2} = m\frac{d^2}{12}\alpha$.

So $\frac{m}{w} = 4$ and so $w = \frac{m}{4} = 10$ kp.

The general solution, through finding S_3

So our question is : $1^3 + 2^3 + 3^3 + \ .\ .\ + n^3 = \Sigma k^3 = S_3 = ?$

Observe that: $(z-1)^4 = z^4 - 4z^3 + 6z^2 - 4z + 1$ and so:

$4z^3 - 6z^2 + 4z - 1 = z^4 - (z-1)^4$. Now let substitute $z = 1, 2, \ .\ .\ .\ , n$:

$4x1^3 - 6x1^2 + 4x1 - 1 = 1^4 - 0^4$

$4x2^3 - 6x2^2 + 4x2 - 1 = 2^4 - 1^4$

$4x3^3 - 6x3^2 + 4x3 - 1 = 3^4 - 2^4$

.

$4n^3 - 6n^2 + 4n - 1 = n^4 - (n-1)^4$. Adding these together:

$4S_3 - 6S_2 + 4S_1 - n = n^4$. And so $S_3 = \dfrac{n^4 + 6S_2 - 4S_1 + n}{4} =$

$$\frac{n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n}{4} = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

Observe that: $n^4 + 2n^3 + n^2 = (n^2 + n)^2$. So $S_3 = \left(\dfrac{n^2 + n}{2}\right)^2 = S_1^{\,2}$ that is:

$1^3 + 2^3 + 3^3 + \ .\ .\ + n^3 = (1 + 2 + 3 + \ .\ .\ + n)^2$.