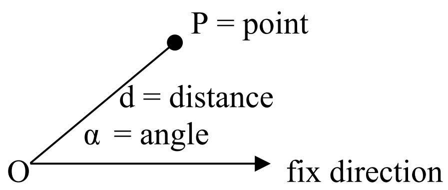
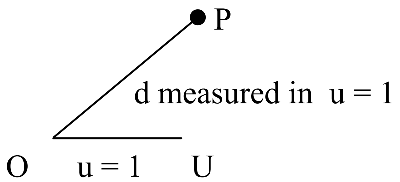
The Three Meanings of Relativity

The first meaning is simply the everyday one and refers to that things can be judged or described differently if they are compared with different other things. For example, an elephant is big relative to a mouse, but small compared to a mountain. A more exact description than "big" can be by mechanical size, like ten metres. But this is even more relative to the unit of metre. Clearly, in kilometres, ten metres is merely a hundredth of a kilometre: 10 m = .01 km. Something similar to this dependence on the units is happening when we use coordinate systems to locate things. The coordinate system itself can be many kind. We can locate a point by its length from a fix O origin, and by the angle of this distance from a fix direction:



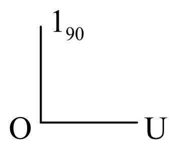
If we want to specify the unit of lengths too, then we can put that unit on the fix direction and thus, we use an other U fix point besides O:



A simple way of writing this location system is by $P = d_\alpha$.

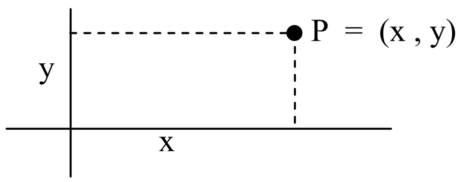
Then, $O = 0_\alpha$ with any α angle, while $U = 1_0$.

A special, important point is 1_{90}, which is the 90° turn of U around O:



This 1_{90} point is usually abbreviated as i and called the imaginary unit of the plane.

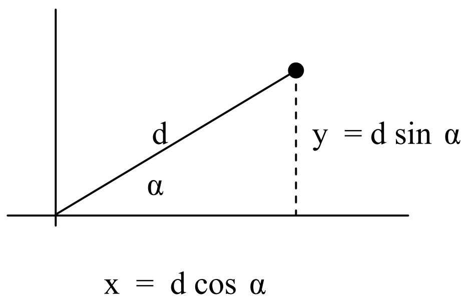
A more frequently used location system is the Descartes coordinate system, that uses two perpendicular x , y coordinate lines:



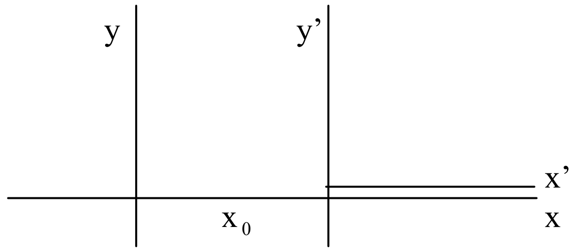
Then we can ask the question how one coordinate system location can be changed to the other. For example, the above used polar and Descartes systems can be transformed as:

$$d_\alpha = (x, y) = (d\cos\alpha, d\sin\alpha)$$

$$(x, y) = d_\alpha = \left(\sqrt{x^2+y^2}\right)_{\arctan\frac{y}{x}}$$

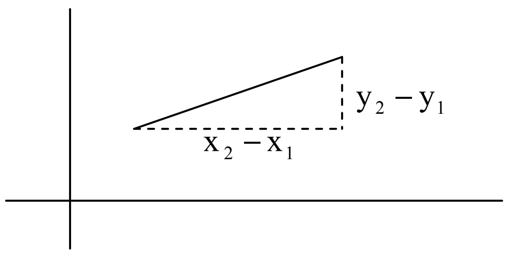


A completely different meaning of transformation is when we use the same kind of system but in different positions, that is the origin and the direction of coordinate lines are different. If for example we only shift the origin along the x-axis with a x_0 distance, then the change in Descartes systems will be only in the x coordinates:



Obviously, for a P = (x , y) point, the change to [x' , y'] is very simple, namely:
$x' = x - x_0$ and $y' = y$

Even in the most complicated situations, the changing of descriptions seem to be following easy from the rules of geometry.
We feel quite natural that even though the location of points is different in different systems, the distances of points must be the same.
In Descartes system, the d distance between two (x_1 , y_1) and (x_2 , y_2) points is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ by the Pythagoras theorem:



Thus, the $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ quantity must be invariant in all systems.

Changing from mathematical descriptions to physical ones, we are less sure what are the changeable and what are the factual invariant properties. Even if we accept that physical properties are more coordinate dependant and thus go along with ideas that size, force, temperature can be relative, we still will be sure that certain facts must remain invariant. For example, one observer might perceive a star's development differently from an other, but if that sun explodes, then this fact itself must be true for the other observer too. So, facts are not relative, but we don't know exactly where facts start from descriptions.
The most important physical quantity beyond mathematics is time. We feel though, that time can not depend on the coordinate systems, or it is the same in all.
A special case of physical coordinate changes is when one system is traveling with a fix speed in an other. First of all, we feel that then the other must travel with the opposite speed in the first. But this can only stand if they use the same units of length and time. Furthermore, we feel that the whole change from one system to the other, is merely a continually increasing shift of locations. If we accept that time is the same in both systems, then this shift is easily calculable for both systems. The extra added invariance of such systems moving with fix speed, is that not only time but all the forces are the same in them too. This of course means that all motions can be

described identically in the moving systems. Strangely, this invariance of descriptions was recognized by Galileo, before even the proper introduction of forces by Newton.
This discovery of Galileo is the second meaning of Relativity, namely as Relativity Principle. Of course, it should be actually called as absoluteness principle, because it claims that something is not variant, that is absolute. But the name simply refers to the relativeness of the coordinate systems. Galileo made his observations in moving boats, but today we can feel them best in trains. When a train is traveling straight with any high speed we can still eat, drink, play, drop objects, just as if the train were standing still.
The third meaning of Relativity, is Einstein's theory of Relativity, which extended the mechanical Relativity principle of Galileo to all physical measurements. The real new feature of Einstein's relativity, was not this natural extension to non mechanical features, rather an extension to a completely illogical, but seemingly almost geometrical situation. Galileo knew perfectly well that if we look out of the boat, we see that we are moving, but the point was that inside its not detectable by mechanical experiments. The same way we see from the train that we move, but it doesn't affect us. Clearly, throwing objects from or into a moving system, will reveal the motion of the system at once. Light coming in seems similar, so the visual experience of our motion fits into this perfectly. Yet, amazingly if we allow a light into our system, but use it merely as a traveling object, not as information about the outer world, then even experiments with these captured lights will remain the same and will not reveal that we are in motion. This becomes surprising if we ask why the old fashioned Galileo Relativity Principle was true at all. Of course, it was not Galileo, only Newton who answered this.

Newton's First Law

The first law of Newton says that bodies keep their speeds if no forces act upon them or if the forces cancel each other and so the total is zero. The second law tells how the bodies change their speeds, when the total forces are not zero. The third law tells that all bodies that cause a force on an other body will receive an equal but opposite force from the other. The force and its counter force are opposite so they cancel each other mathematically, but not in the sense we mentioned in the first and second law where we were talking about only the forces that act upon one single body. Finally, the fourth law of Newton is his special description of the gravitational force.
Newton was able to derive Kepler's laws of the planetary motions from his laws and thus proved without a doubt that he indeed discovered the fundamentals of physics.
The strange about the first law is that it mathematically follows from the second, if we use zero force. So why did he start with the special forceless situation? In one sense, it was to set the mind of the reader to a new understanding. Indeed, before Newton the naïve belief was that a force causes a motion only for a while, because things slow down and stop naturally. Newton was the first to realize that the slow down and the stop is caused by appearing forces of frictions. Everybody knew that oiling surfaces will lessen frictions but nobody realized that if there were no frictions at all, then motions would go on forever. Today, this idea is served to us on a silver plate, because the images of the empty forceless cosmos is revealed in documentaries and even in science fiction films. But as usual, if something is served on a silver plate, then we lose our work in the understanding. And indeed, the image of objects freely moving in empty space is combined with the forcelessly floating objects in the space capsules. And yet, these two are two completely different situations. In the spaceship that orbits the earth, the earth's gravitation is definitely there. The astronauts are only floating because they are orbiting exactly as the ship itself. Indeed, even in a falling elevator, we would feel the same weightlessness. By the way, this artificial weightlessness, is not perfect forcelessness. After all, all forces direct toward the

center of the earth, and thus they are not completely parallel, but have a very small angle. If for example, two balls float in a falling elevator, then after a long enough fall, they would get a little bit closer to each other. This would seem for a falling observer as some unexplainable small force attracting the balls. The gravitation of the balls is even smaller than this imaginary force, so that can't be an explanation. Just how confused people were about the new physics of Newton, even hundreds of years later, can be seen from the mistakes Jules Verne made in his famous book, Trip To The Moon. First of all, he didn't know the principle of rockets, so he assumed that such trip would be made by a bullet space capsule shot out from a big cannon. But his biggest mistake was that he ignored the artificial weightlessness and thought that the astronauts would only become weightless when the spaceship reaches the point between the moon and the earth where the two gravitations cancel each other. Strangely, it didn't occur to him that the sun's gravitation is still working, so if a weightlessness appears at the critical point, then its already an artificial weightlessness, relative to the sun. So as we see, the first law didn't help to put people's mind in the correct vision.

Today the first law is explained as not even a law, but rather a definition to select those systems where the other three laws are valid. I don't remember to find any sign of this intention in Newton's original Principia Mathematica, but it's a hard reading, so if somebody can point me out this tendency, I'll be happy to hear about it. Anyway the idea to select the systems where the Newton Laws are valid, is important because they can't be true in any system, as we saw in the case of the falling elevator.

The real problem with the first law is that it claims some speeds to remain the same, but doesn't tell in what system were those speeds meant in the first place. Clearly, only in systems without hidden forces could the speeds remain the same. But then, to use the first law as selection to exclude systems with hidden forces is useless because we are in a self defining cycle. But the situation is not as bad as seems. The reason is that the first law claims <u>all</u> speeds to remain the same. So, throwing bodies in all directions and with all different speeds, the actuality of which speeds are the real disappears and also if the system had some hidden forces it would show in some directions and speeds. For example, we saw above that in a falling lift, the gravitation doesn't completely disappear because objects fall closer and closer to each other, so in the lift, they don't stay still, as the first law would require. If we regard the whole earth and take into consideration its gravity, then we'd have to cancel somehow the gravitation in order to test the conditions of the first law. A simple way to do this is by only allowing objects to move on flat surfaces perpendicular to the radius of the earth. Then, the surface compensates the gravitation. A book on a table is in rest because the table cancels the weight of the book. Of course, it still could slide sideway, but that is stopped by the friction. On an icy surface, we slide forcelessly, except of course the still existing friction of the skate. But there is an other hidden force still on these surfaces. Indeed, we forgot about the daily spinning of the earth. When the ice skater is standing still then actually she is orbiting with the earth's surface. If the earth's gravitation wouldn't keep her on the ice, she would be traveling at every moment on a straight line with her last speed. This line would leave the curving of the earth. Even though the gravitation is much bigger, this effect of the spinning, still takes a little bit off the weight of the skater. The biggest of this weight loss is on the equator, because that's the largest spinning circle. As we go, north or south, the spinning circles are getting smaller and at the north and south pole, the full gravitational weight is experienced. Not only are the circles getting smaller, but their plane is not containing the radius which is the true direction of the gravitation. For a standing body, this doesn't make much difference, but if the skater is in motion, then strangely beside the minute loss of weight, an additional minute force appears in the surface sideway to her motion. This minute sideway force is enough to direct the flushing of the toilets or sinks in one direction on the northern hemisphere, while oppositely on the southern.

The best demonstration of this sideway force though is the famous Foucoult Pendulum. This is a very long pendulum with a sharp weight that scratches lines in sand. Amazingly, in a full day the scratchings go around and make a full turn. In reality, it was the sand that made a full turn, but in the system of the earth, an imaginary force had to move the pendulum sideway.
The aboves clearly show that the earth is a complicated system with many hidden forces, so we might think that Newton's First Law is a proof of the fact that the earth goes around the sun and not oppositely. The situation is more complicated though. First of all, our sun is in the milky way galaxy, which is turning, so the sun is in motion itself. Secondly, even for two bodies, picking the bigger one as the center is not perfect either. The moon is not circling the earth, rather they both go around the common center of gravity, which is inside the earth, but not at the center. This weight center is that orbits the sun. All these complications bring back an older argument between Galileo and the church. The oversimplified view is that Galileo was forced to keep his mouth shut and not join Kepler in standing up for the new sun centered view of the world. Cardinal Bellarmino sent a letter to Galileo's friend and expressed an amazing opinion about the whole quarrel. He said that the church has absolutely no problem with the mathematical description of the planets going around the sun in whatever orbits. In fact, if it is more accurate for calendrical calculations, its appreciated. After all, all motions are relative anyway. But to claim that in reality the sun is standing and not the earth, is not only against the scriptures, but contradicts the common sense of simple people.
Newton probably saw deeper than Kepler or Galileo, in this problem of what is standing and what is moving. If the sun is moving itself, then what is the absolute standing system in which the whole universe is moving. On one hand, he believed in such absolute system, on the other he almost realized that it can't exist.

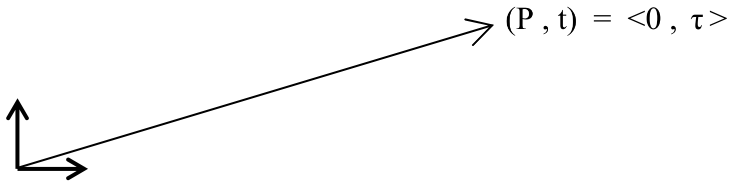
Light

It's quite logical that after discovering the complete mechanics, Newton was trying to apply this mechanics as far as possible. Unfortunately, he was unable to derive the laws of light reflections and colors from mechanical principles. Thus, it seemed that in this single fight, he lost against Huygens, who believed light to be a wave. This is a strange contradiction as I just mentioned above, because if light is a wave, it must travel in a medium that exists in empty space too. Indeed, the sun's light comes to us through empty space. But then, if there is such medium of the empty space, then it proves that there is an absolute standing system in which the universe is embedded. So in a sense, Huygens' opinion would've been the perfect ground for his mechanics. On the other hand, if light is merely a ray of tiny particles that travel like bullets, then there is no need for absolute medium and so an absolute coordinate system can merely be a geometrical abstraction without matter.
Light turned out to be more complicated than anything before and thus, made both Newton and Huygens winners. Light also made history repeat itself namely, the relationship of Kepler and Newton replayed again between Maxwell and Einstein. Kepler discovered the laws of planets but couldn't put them in the real world and only Newton was able to find the bigger picture. Similarly, Maxwell discovered the laws of light but couldn't put them in the old physics. Einstein realized that Maxwell's laws are the correct ones and rather the whole old physics had to be rewritten.

Self Time

A (P , t) event is reachable if $v = \frac{|P|}{t} < c$.

Indeed, if (P , t) is reachable then one can reach (P , t) from the $(0 , 0)$ initiation with the fix v speed. If this motion is done by a $\langle\ \rangle$ system, then in that system (P , t) is an event in the origin at a τ time. In other words, $(P , t) = \langle 0 , \tau \rangle$:
This τ time is called the "self " time of (P , t). Indeed, if $\langle\ \rangle$ is a moving clock, then τ is the actual time that this shows when it gets to (P , t).



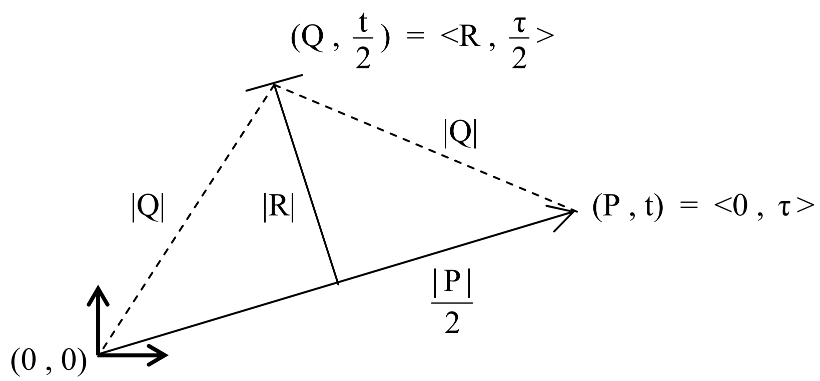
The question why a moving clock would change its pace is ignored here! But we have to realize that maybe even this question itself is meaningless and a more realistic question is whether such changes are factual or merely descriptive. Indeed even though the traveling clock can find other standing ones nearby that reveal that it is showing different time, this only makes sense if we are sure that all standing clocks can be synchronized. We'll deal with this soon. Right now we should be amazed that instead of starting from distances we jump right into time and try to find a formula for the self time. This will be the basic equation to find the rules of all relativistic changes.

The fundamental assumption is that the speed of light is the same c value both in $(\ )$ and $\langle\ \rangle$. But there will be other "hidden" assumptions too.

We'll give two derivations! One is the original approach of Einstein by regarding light reflections perpendicular to the motion of the $\langle\ \rangle$ system, while the second, only uses same directional reflections but it's a bit more complicated:

For better visualization, we can imagine that the $(\ )$ system is our earth. An alien space ship is the $\langle\ \rangle$ system that passes by earth when the times are set to 0 in both systems. Then the $(P , t) = \langle 0 , \tau \rangle$ event is that the space ship reaches Jupiter.

The space ship has small companion ships traveling with the same speed at different distances. When the space ship passes by earth, it sends a light signal perpendicular to its path. One of its companion ships reflects this light back and it returns to the main ship exactly when it reaches Jupiter:



$$t = \frac{2|Q|}{c}$$

$$\tau = \frac{2|R|}{c} = \frac{2\sqrt{|Q|^2 - \left(\frac{|P|}{2}\right)^2}}{c} = \sqrt{\frac{4|Q|^2}{c^2} - \frac{|P|^2}{c^2}} = \sqrt{t^2 - \frac{P^2}{c^2}}$$

The t time at P, was obtained as a reflected time of light. This is in accordance with our assumption that the speed of light is the same in all systems. One objection could be that the reflection was done by the moving mirror of a companion ship. In reverse, if we imagined standing mirrors in the ( ) system, then we could object the exact reflection in the $< >$ system. To avoid this problem we can synchronize the clocks of a system by direct light signals too. At (0 , 0), light signals are sent in every direction. When the clock at P receives this, it will know that its clock must be synchronized not to 0, rather to $\frac{|P|}{c}$. In fact, we don't even need clocks at all places, because the origin can continually send the time which can be modified at P with $+\frac{|P|}{c}$.

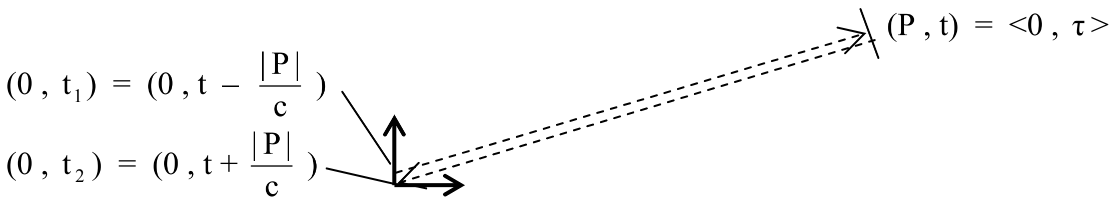
Lets imagine this in our next, second derivation.

When the $< >$ space ship gets to P = Jupiter, then the t_1 time signal of the origin = earth, is received too. As we said, $t = t_1 + \frac{|P|}{c} \rightarrow |P| = c\,(t - t_1)$.

Suppose Jupiter reflects this t_1 back to earth and it returns at t_2.

The returning light traveled $t_2 - t$ time, so again $|P| = c\,(t_2 - t)$.

From these, t_1 and t_2 can be expressed as $t - \frac{|P|}{c}$ and $t + \frac{|P|}{c}$:



Now we imagine that a same directional second ship also approached earth, ahead of the first, but with half the speed. This ship can be denoted as a [ ] system.

Also, lets assume that the two space ships arrive to earth at the same time. Here, they both set their clocks to 0. They all keep on traveling with their fix speeds, but of course now the second ship is behind the first. In fact, this ship will see the first going forward and the earth moving back with the same $u = \frac{v}{2}$ speeds.

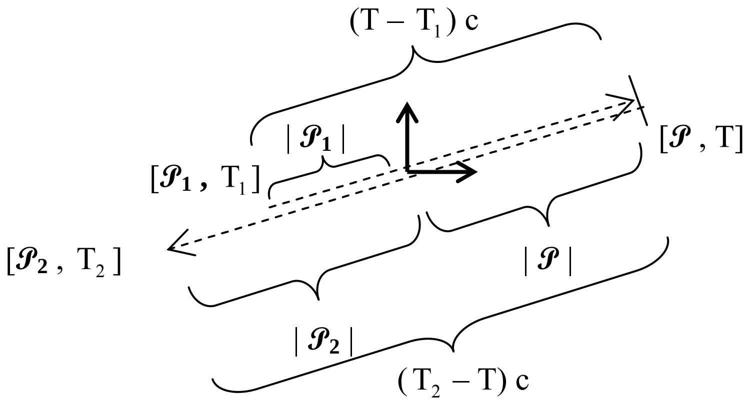
The light signal, mirroring and return can be looked from the [ ] system as:

$(0 , t_1)$	- - - - - - -▶	(P , t)	- - - - - - -▶	$(0 , t_2)$
||		|| $<0 , \tau>$ ||		||
$[\mathscr{P}_1 , T_1]$		$[\mathscr{P} , T]$		$[\mathscr{P}_2 , T_2]$

We'll assume that when time at an origin of a system is measured in an other system, then the new time is merely a linear multiplication by a factor that depends only on the relative speeds. If for the u relative speed this factor is λ, then,

$T_1 = t_1\,\lambda \;,\; T_2 = t_2\,\lambda \;,\; T = \tau\,\lambda$

That's why this middle moving [ ] system was applied because it could connect ( ) and < >. Furthermore, in [ ] the light signal, reflection and return is easily calculable too as:



$$|\mathcal{P}_1| + |\mathcal{P}| = T_1\,u + T\,u = (T - T_1)\,c \rightarrow T_1 = T\,\frac{c-u}{c+u}$$

$$|\mathcal{P}| + |\mathcal{P}_2| = T\,u + T_2\,u = (T_2 - T)\,c \rightarrow T_2 = T\,\frac{c+u}{c-u}$$

$$\Bigg\} \quad T = \sqrt{T_1\,T_2}$$

Thus,

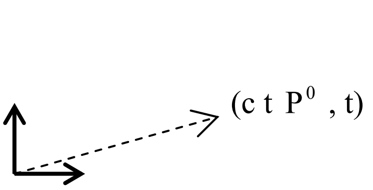
$$\tau = \frac{T}{\lambda} = \frac{\sqrt{T_1\,T_2}}{\lambda} = \frac{\sqrt{t_1\,\lambda\,t_2\,\lambda}}{\lambda} = \sqrt{t_1\,t_2} = \sqrt{\left(t - \frac{|P|}{c}\right)\left(t + \frac{|P|}{c}\right)} = \sqrt{t^2 - \frac{P^2}{c^2}}$$

Complement Of Unreachable Event

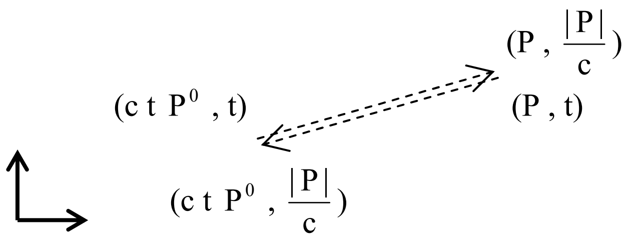
If $\frac{|P|}{t} = c$ then (P, t) is light event.

If $\frac{|P|}{t} > c$ then even a light sent from $(0, 0)$ can not reach (P, t).

If P^0 denotes the unit vector of P, then the place in between 0 and P where the light reaches at t is $c\,t\,P^0$:

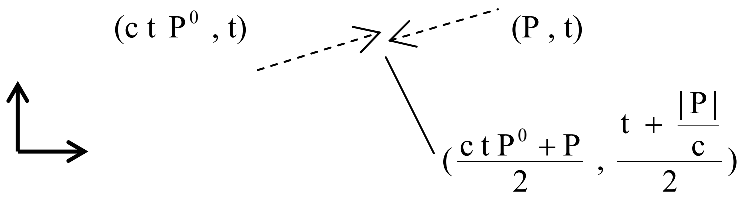


If we allow this light to continue towards P, it will only get there later, namely at $\frac{|P|}{c}$ time. If we also send a light back towards the origin from P at t, then this gets to $c\,t\,P^0$ exactly the same time as the continuing one reaches P, that is at $\frac{|P|}{c}$:



This $(c\,t\,P^0 , \frac{|P|}{c})$ is called the complementer of (P , t).

The continuing and the back traveling lights will meet at the middle of the $c\,t\,P^0$ and P places, that is at $\frac{c\,t\,P^0 + P}{2}$ and at the middle of the t and $\frac{|P|}{c}$ times, that is at $\frac{t + \frac{|P|}{c}}{2}$:



Thus, the complementer of (P , t) can also be regarded as the continuing doubling of the light trip from (P , t) to this middle light crossing. This can be used to prove that the complementer of (P , t) is independent of the system we use. So the title "complementer of unreachable event" was meaningful after all. First we can see that the light crossing from $(0 , 0)$ and (P , t) remains the similar light crossing in an other system. Indeed, the fix speed of light and the shortest distance being on a line guarantees this. Then the reflection of (P , t) to this light crossing event is again a continuation of lights, so can be expected to remain the same.

<u>Poincare Minkowski Invariance</u>

If (P , t) is unreachable, then it doesn't have self time, and indeed (P , t) couldn't be written into the $\tau = \sqrt{t^2 - \frac{P^2}{c^2}}$ self time formula, because under the square root, negative would appear. Accepting imaginary numbers of course, allows the substitution and most interestingly:

$$\tau = \sqrt{t^2 - \frac{P^2}{c^2}} = \sqrt{-1}\,\sqrt{\frac{P^2}{c^2} - t^2} = \sqrt{-1}\,\sqrt{\left(\frac{|P|}{c}\right)^2 - \frac{(c\,t\,P^0)^2}{c^2}} = \sqrt{-1}\;\tau_{comp}$$

So the self time of an unreachable (P , t) is the imaginary version of the real self time of (P , t)'s complement. This is nice, but has no physical meaning at all! A new real meaning will emerge for using unreachable events in the self time formula, if we discover a new meaning for the old reachable cases first. This meaning is that since the self time is independent of the ( ) system, thus the $\sqrt{t^2 - \frac{P^2}{c^2}}$ value must be the same in all systems. For unreachable events, this should be true too because the complement is not system depending. So, $\sqrt{-1}\;\tau = \sqrt{\frac{P^2}{c^2} - t^2}$ is invariant.

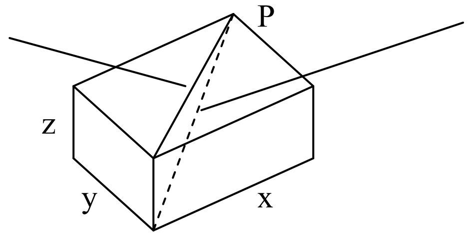
Observe that if (P , t) is reachable and τ is its real self time, then the right is imaginary, while if (P , t) is unreachable, then the right is real and so is the left too, because τ is imaginary. Then of course, multiplying both sides with c:

$\sqrt{-1}\ c\ \tau = \sqrt{P^2 - c^2 t^2} = \sqrt{x^2 + y^2 + z^2 + (\sqrt{-1}\, c\, t)^2}$ is invariant too.

The imaginary variation for the self time was a juggling. But now in this light speed multiplied version, not only the left self time, but also the right real time is measured in the same $\sqrt{-1}$ c manner. This is a big coincidence, plus on the right a fourth dimensional Pythagorean distance appeared with time measured in this manner.

Let's remember that $\sqrt{x^2 + y^2 + z^2}$ is the distance of an (x , y , z) place from the origin. Indeed this can easily be obtained with repeated use of the simple two dimensional Pythagoras theorem:

$\sqrt{x^2 + y^2}$



$|P| = \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$

In classical physics, the $\sqrt{x^2 + y^2 + z^2}$ distance from the origin is obviously not invariant only the $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ distance between two points. So the new four dimensional invariance of $x^2 + y^2 + z^2 + (\sqrt{-1}\ c\, t)^2$ is actually a simpler invariance than the classical.

Allowing time as a fourth dimensional "complication", we gained a surprising simplification, namely that single points or events already possess invariance not only pairs.

An even more "grand" view of this invariance comes about if we go back again to three dimension and ask what would be the meaning of such invariance of a distance from the origin. Obviously a mere turning of the distance into a new position:

Or in reverse this means that the three coordinate axes are merely repositioned or turned. Such turning of a coordinate system was already an option in classical physics but instead we usually kept the coordinates parallel and emphasized the seemingly much more important fact that the two system can travel from each other. The amazing result is that the invariance that exists between traveling systems melts into the primitive non traveling that is merely turned system relations! Indeed by going four dimensional: All invariance is merely turning!

These four dimensional meanings were observed by Minkowski after Einstein's Special Relativity, but also earlier by Poincare without realizing the physical implications later discovered by Einstein. Amazingly, Poincare even used the word relativity and presented the $E = mc^2$ formula as a weird mathematical possibility.

Einstein did not know about Poincare's results and so he started from scratch. Still when his attention was drawn to Poincare's earlier results, he appreciated those and made a remark of them in his original publication. Poincare was also among the firsts to spread Einstein's results but never tried to take credit for those. Sadly, later when Einstein became famous, he never mentioned Poincare's results or even uttered his name any more!

Linear Relativistic Transformations

We'll assume that the change from (x , y , z , t) to [x' , y' , z' , t'] can be described by linear equations:

$$x' = \alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 t$$
$$y' = \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2 t$$
$$z' = \alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3 t$$
$$t' = \alpha_t x + \beta_t y + \gamma_t z + \delta_t t$$

We already know the Poincare Minkowsi invariance, that is that the transformation will be a four dimensional turn but instead of starting with that, we'll use the initial three dimensional turn of the systems. If there were no such turn that would mean that if [ ] travels in ( ) with a V velocity, then in reverse ( ) travels in [ ] with – V . So to assume an initial turn is simple, by allowing that ( ) travels in [ ] not with – V rather with an arbitrary W. Of course the actual speed value of the two velocities should be the same that is: $|W| = |V|$. Allowing W to be completely independent of V means that even the units of length and time are different in the two systems. A perfect melting of the initial turn and used units into the general turn would be, if the choice of V and W could completely determine the general turn. That's exactly what we'll achieve:

The origin of the ( ) system moves with $W = [w_1 , w_2 , w_3]$ in the [ ] system. That is, $x = y = z = 0 \rightarrow x' = w_1 t' , y' = w_2 t' , z' = w_3 t'$. Thus:

$$t' = \delta_t t \rightarrow \frac{t'}{t} = \delta_t$$ we can use this for the following three:

$$w_1 t' = \delta_1 t \rightarrow \boxed{\delta_1 = w_1 \delta_t}$$
$$w_2 t' = \delta_2 t \rightarrow \boxed{\delta_2 = w_2 \delta_t}$$
$$w_3 t' = \delta_3 t \rightarrow \boxed{\delta_3 = w_3 \delta_t}$$

The origin of the [ ] system moves with $V = (v_1 , v_2 , v_3)$ in the ( ) system. That is, $x' = y' = z' = 0 \rightarrow x = v_1 t , y = v_2 t , z = v_3 t$. Thus:

$$0 = \alpha_1 v_1 t + \beta_1 v_2 t + \gamma_1 v_3 t + \delta_1 t$$
$$0 = \alpha_2 v_1 t + \beta_2 v_2 t + \gamma_2 v_3 t + \delta_2 t$$
$$0 = \alpha_3 v_1 t + \beta_3 v_2 t + \gamma_3 v_3 t + \delta_3 t$$

Dividing each with t and using the previous equations:

$$\boxed{\begin{aligned} v_1 \alpha_1 + v_2 \beta_1 + v_3 \gamma_1 + w_1 \delta_t &= 0 \\ v_1 \alpha_2 + v_2 \beta_2 + v_3 \gamma_2 + w_2 \delta_t &= 0 \\ v_1 \alpha_3 + v_2 \beta_3 + v_3 \gamma_3 + w_3 \delta_t &= 0 \end{aligned}}$$

Calculating $x'^2 + y'^2 + z'^2 - c^2 t^2$ from the equations and using the Poincare Minkowsi invariance we get ten more equations:

x^2-s coefficient is	$\sum \alpha_i^2 - c^2 \alpha_t^2 = 1$
y^2-s coefficient is	$\sum \beta_i^2 - c^2 \beta_t^2 = 1$
z^2-s coefficient is	$\sum \gamma_i^2 - c^2 \gamma_t^2 = 1$
t^2-s coefficient is	$\sum \delta_i^2 - c^2 \delta_t^2 = -c^2$
2 xy-s coefficient is	$\sum \alpha_i \beta_i - c^2 \alpha_t \beta_t = 0$
2 xz-s coefficient is	$\sum \alpha_i \gamma_i - c^2 \alpha_t \gamma_t = 0$
2 yz-s coefficient is	$\sum \beta_i \gamma_i - c^2 \beta_t \gamma_t = 0$
2 xt-s coefficient is	$\sum \alpha_i \delta_i - c^2 \alpha_t \delta_t = 0$
2 yt-s coefficient is	$\sum \beta_i \delta_i - c^2 \beta_t \delta_t = 0$
2 zt-s coefficient is	$\sum \gamma_i \delta_i - c^2 \gamma_t \delta_t = 0$

Thus, we obtained exactly 16 equations for the 16 unknown coefficients.
These linear equations together are also called the coordinate transformation.

x Directional Relative Motions

If the two coordinate systems are not turned and use the same units, then $W = -V$.
If we also assume that V is in the x axis, then $V = (v, 0, 0)$, $W = [-v, 0, 0]$.
In other words, $v_1 = v$, $v_2 = v_3 = 0$, $w_1 = -v$, $w_2 = w_3 = 0$. Thus:
From the first framed equations: $\delta_1 = -v\,\delta_t$, $\delta_2 = \delta_3 = 0$
From the second framed equations: $\alpha_1 = \delta_t$, $\alpha_2 = \alpha_3 = 0$
Then the third framed equations can be solved giving the final result as:

$x' = \alpha x - v\alpha t$
$y' = y$
$z' = z$
$t' = \alpha t - \frac{v}{c^2}\alpha x$

Where $\alpha = \frac{c}{\sqrt{c^2 - v^2}} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$

We'll show a direct way to get α.
Assuming a linear $(x, t) = [x', t']$ and the conservation of y and z:

$x' = \alpha x + \beta t \qquad y' = y \qquad z' = z$
$t' = \gamma x + \delta t$

$(v, 0, 0, 1) = [\alpha v + \beta, 0, 0, \gamma v + \delta] = [0, 0, 0, \tau] \rightarrow \beta = -v\,\alpha$, $\gamma = \frac{\tau - \delta}{v}$

$(0, 0, 0, \tau) = [\beta\,\tau, 0, 0, \delta\,\tau] = [-v, 0, 0, 1] \rightarrow \tau = \frac{1}{\delta}$, $\beta = -\frac{v}{\tau} = -v\delta$

Thus, $\delta = \alpha$, $\tau = \frac{1}{\alpha}$ and $\gamma = \frac{\frac{1}{\alpha} - \alpha}{v}$. Finally:

$(0, c, 0, 1) = [\beta, c, 0, \delta]$ = light event $\rightarrow \sqrt{\beta^2 + c^2} = c\,\delta$. Squaring and using $\beta = -v\,\alpha$, $\delta = \alpha$, we get: $v^2\alpha^2 + c^2 = c^2\alpha^2 \rightarrow \alpha^2 = \frac{c^2}{c^2 - v^2}$.