The Contact Restriction Problem

Definition and finite case

An f function assigning to all members of an A set a subset of A is a contact function.
Unlike knowing someone that feels symmetrical, this merely means a one sided relation like indeed making a contact toward someone. A person can try to contact everybody which is not that easy but still if achieved automatically implies something that we will focus on.
Namely, whether there is a contactless pair meaning that none of them contacted the other.
So we can introduce an r restriction value on how many contacts one allowed to make.
Then the contactless pair problem becomes more interesting. Actually a new angle enters too.
Namely, the question of chances, that is how probable a randomly chosen pair to be contactless.
That things are surprising is shown by the following simplest case:
A is a three element set and we make the seemingly very strong $r = 1$ restriction. And yet with making the contacts as a three member cycle, we will have no contactless pair. This shows that smartly making the contacts we can avoid such pairs. But still, with A having more n members and using a relatively small r we can make sure that there will be contactless pair.
Jumping over the combinatorial questions of the finite situations, we can ask what happens if A is an infinite set.

The simple restriction failure paradox

The simplest question is this: In an infinite A set restricting the number of contacts to be finite will we have contactless pair for sure? Amazingly, the answer is no! Indeed:
Let A be the set of the natural numbers and every number should contact the smaller ones.
But the real paradox is that this is universal!
So in every infinite A set with a similar $f(x) < A$ contact restriction we can make a smart f with which no contactless pair will be in A.
The construction of f requires to make a minimal well-ordering of A where all beginnings have less infinity than of A. Then order to every x member the beginning before x.

Strengthening the contact restriction

The obvious idea is to allow even less contacts, that is make r "way smaller" than A.
Here comes in an other use of the above natural number set example where there was no previous size to the $|A| = \omega$ so called countable first infinity.
The next infinity ω_1 is different because it has ω as previous. Then ω_2 is such again and so on but then the first infinity after this sequence is ω_ω which is like ω was, a limit infinity.
So to make our treatment universal, we must regard not merely a single r infinity restriction rather an R set of infinities.
The "way smaller" than just being less than $|A| = \omega_\alpha$ could then mean that there is an ω_β infinity "between" R and $|A|$. That is, for all $\omega_\gamma \in R$ having that $\omega_\gamma < \omega_\beta < \omega_\alpha$.
Amazingly, with this we can now show quite easily that we succeeded and so there is no smart f that can avoid a contactless pair if f is obeying our new restriction that $|f(x)| \in R$.
Indeed, let B be a subset of A with $|B| = \omega_\beta$ and let $f[B]$ denote the combining of all the $f(x)$ sets for $x \in B$. Then $|f[B]| \leq |B|\,\omega_\beta = \omega_\beta\,\omega_\beta = \omega_\beta < \omega_\alpha = |A|$.
Thus $A - f[B]$ is a non empty set, that is it has some a member.

Also, for all x we have that $|f(x)| < \omega_\beta = |B|$ and thus for all x the $B - f(x)$ set is non empty, that is there is some y member of it. In particular, for $x = a$ too, there is a $b \in B - f(a)$.
So: $a \in A - f[B]$ and $b \in B - f(a)$ which imply that $a \notin f(b)$ and $b \notin f(a)$.

Combined form of the paradox and the success

The above two results together mean the following theorem that Sierpinski recognized before the War already:

Let R be a set of infinities and ω_α an infinity bigger than any of the ones in R.

If there is an ω_β infinity between R and ω_α, that is bigger than any one in R but smaller than ω_α then and only then:

For every f contact function with $|f(x)| \in R$ restriction in an A set with $|A| = \omega_\alpha$ there is a contactless a , b pair in A.

The "then" is exactly our above restriction strengthening result.
The reverse is exactly the contactless pair avoiding f construction in our paradox before.

The Continuum Hypothesis

Cantor after proving that the set of infinite decimals is bigger than the set of naturals raised the obvious question whether there is an infinity between.
He believed that there isn't, that is ω_1 is the infinity of the decimals.
This became called the Continuum Hypothesis and remained open till the sixties.
Paul Cohen finally proved that it is unprovable from the present axioms of Set Theory.
Chris Freyling in 1986 suggested a new so called Axiom Of Symmetry that would prove the falsity of the Continuum Hypothesis.
By our above results it would be enough if using R as $\{\omega\}$ we could show that the contacts with this restriction always imply the existence of a contactless pair.
So he claimed that if we assign to every point of an interval a sequence of points there, then there has to be a pair of points that neither is in the sequence assigned to the other.
A sequence is miniscule relative to an interval but any infinity smaller than the full interval is such by the same vision. So everything he claims by the sequence assumption could be repeated by being just ω_1 if the full set of points say would be ω_2.
What's worse is that he used measure as argument for the sequence being so miniscule.
As already Cantor showed nil measured sets can have the same infinity as the full set of the interval. These are the famous Cantor sets.
What's even worse is that Bernstein's non measurable construction opened a not well known side problem about the possibility of very dense sets with in-between infinities.
The "Axiom Of Symmetry" would say nothing about these. So it doesn't connect just proves what Freyling wanted to get.

A simple application

I will raise a seemingly geometrical problem that can be solved by our results.
Suppose we have an f function that assigns for every P point of the line finite many points:
$f(P) = \{ Q_1 , Q_2 , . . . , Q_n \}$. Prove that there have to be two P , R points so that neither is among the f values of the other. We'll actually tell how to find such P , R
Take any $P_1 , P_2 , . . .$ sequence of points on the line. Combine all their Q values that will be again just a $Q_1 , Q_2 , . . .$ sequence. Since the full line is non sequencable, there is some R point outside these Q-s. Since $f(R)$ is finite there has to be some P_N not in $f(R)$.
This P_N and R should be our pair. We already chose P_N not in $f(R)$ so this half is true.
Also $f(P_N)$ is a subset of the Q sequence we created and so R is outside this too.