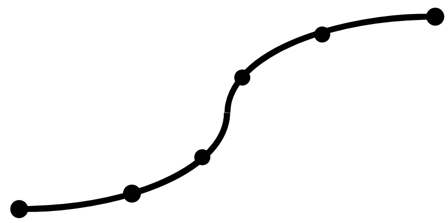
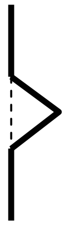
The Five Simple Facts About The Mandelbrot Set

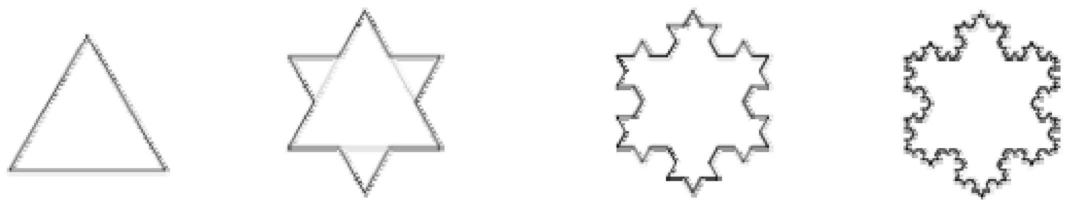
To approximate a curve seems natural by locating more and more known points of the curve, closer and closer to each other, and connecting the neighboring ones with straight lines.



Surprisingly, we can make a “curve” where this method definitely fails. The idea is to start with a single interval and make a bump or sharp detour on the middle, say replace the middle third with two sides on a left or right triangle:



Then we can repeat this for the new 4 straight segments. And then, again and again.
We could even choose the left or right side of the detours by some rules.
The same idea can be applied to special closed curves or boundaries of a point set.
For example, we can start with a triangle and apply our previous line detours to all three sides.
If the three first detours are all done outward it will give a six-star or Star of David.
Continuing, we get the shape of a snow flake.

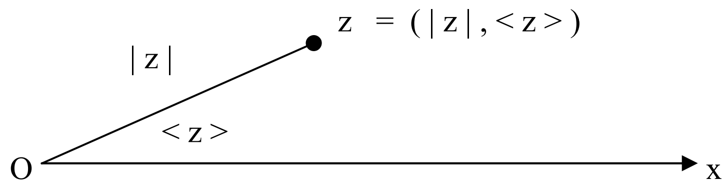


For such infinitely wiggly boundaries, another interesting question becomes what the actual area of the bounded point set will be.
The big surprise turned out to be that quite oppositely, we can define some bounded point sets and their boundaries become wiggly by themselves, purely from the definition of the point set.
In fact, these boundaries are wiggly much more wildly than we could make it by ourselves.
The even more surprising fact is that to obtain such totally unpredictable boundary line can happen with the set itself being defined by a primitive rule.
The trick is that this primitive rule does not directly give the points, rather we apply the rule repeatedly, and then watch how the resulting z_1 , z_2 , z_3 , . . . sequence of points behave.
Namely, whether they go to infinity or not, that is remain in a bounded region.
The false naive assumption was earlier that if a continuous transformation creates z_2 from z_1 then z_3 from z_2, and so on, then this would make the nearby points to z_1 go nearby to z_2 then nearby to z_3, and so on, and thus the divergence or boundedness would be similar too.

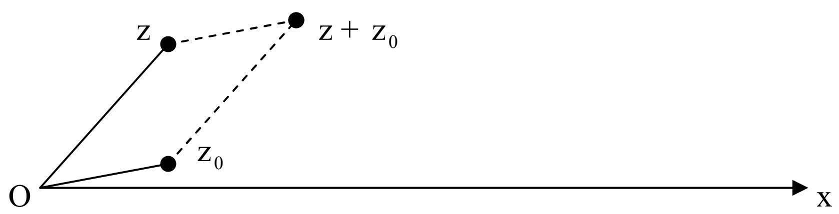
This however is not true! Continuity indeed implies that a fix finite many repetition creates points nearby, but this doesn't imply that the divergence can not happen for the full sequence.
So, closer and closer points can still have different end results in regard of divergence or boundedness. Only the actual examples make this affair clear.

A mystical spin was then attached to all this, as if the points had a secret unruled world.
The continuity of point sets is indeed mysterious, but these finer and finer wiggles on the borderline are not mysterious at all. The fact that they never repeat perfectly, that is don't become periodic, is perfectly natural. An analogue will explain this:
The rational numbers are the periodic decimals. An irrational, like $\sqrt{2}$ has totally "unpredictable digits" if it is meant by non repeatedness. Yet, these digits are very predictable.
Here, in two dimension, the non periodicalness is simply more vivid visually.

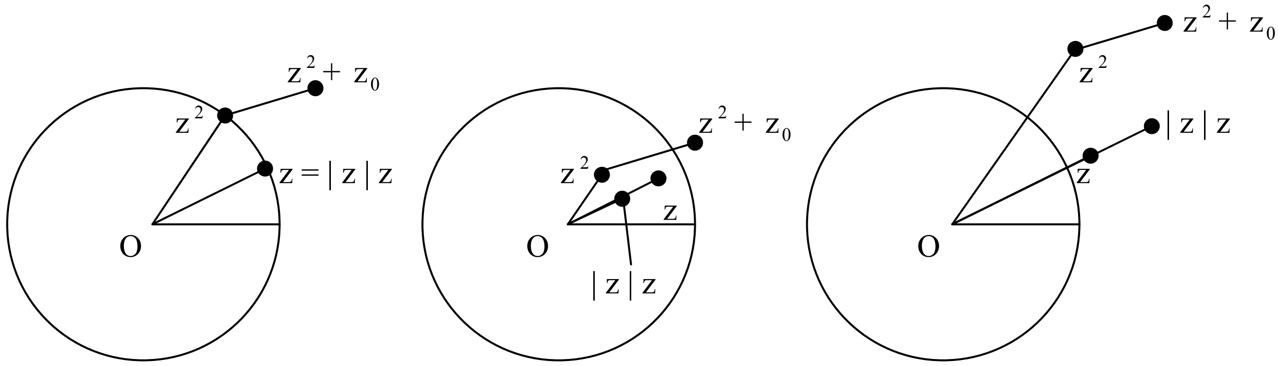
The mentioned primitive transformation from z_1 to z_2 then z_2 to z_3 and so on, can be regarded in three steps. The first is a stretch, that is the alteration of the distance from a fix O origin, but keeping the radial line from O. The second will be a rotation around O and the third is a shifting. To use our transformation formally, the best is to use not a Descartes, but rather a polar coordinate system.
Here, the z points have two components, their length from O denoted as $|z|$ the absolute value and the angle of z's radial from O to a fix horizontal x axis, denoted as $<z>$.
So, $z = (|z|, <z>)$



The multiplication of z with a c real number will be defined as $c z = (c|z| , <z>)$.
Thus, we'll keep the angle and only apply a c stretch.
Such fix c stretch is not enough to produce the interesting wiggles and rather we need a second order or squared stretch. So this actually means to form: $|z| z = (|z|^2 , <z>)$
The beauty of this is that for $|z| = 1$, that is on the unit circle, this stretch does nothing, but inside we compress z toward O while outside, indeed it stretches more and more outward.
But in itself, this is still not enough. We need the further two steps.
First, a linear, that is first order turn, and then a fix, that is 0 order shift.
The linear turn would mean that $<z>$ is changed to $c<z>$.
We'll stick with the simplest $c = 2$ choice.
This also fits into our notations if we introduce the point multiplications in general as:
$u v = (|u|, <u>)(|v|, <v>) = (|u||v|), (<u> + <v>)$
In short, we multiply the lengths but only add the angles.
Then, the $z = (|z|, <z>)$ point squared is exactly the one we'll use as second step, because:
$z z = (|z|, <z>)(|z|, <z>) = (|z|^2 , 2<z>)$
Finally, the fix shift fits again into our notation as a $+ z_0$ addition if we use vector addition.
This means that the $O z_0$ radial is moved parallel and same directionally to the point to which we want to add z_0 and then the end of this moved parallel version of z_0 will be the sum:



Thus, our transformation written in our introduced $z \to (z^2 + z_0)$ or so called complex notation, but broken into the three actual geometrical steps and shown on, in and outside the unit circle will look like this:



We apply this to a starting z_1 to form $z_2 = z_1^{\,2} + z_0$ then we form $z_3 = z_2^{\,2} + z_0$ then $z_4 = z_3^{\,2} + z_0$ and so on. The turns and shifts can play against each other and this creates the crucial steppings in or out of the unit circle.

The Julia set collects all those z_1 points that if used as start will bring about a bounded sequence. So: Julia $(z_0) = \{ z_1 \ ; \ z_2 , z_3 , \ldots$ are bounded $\}$

The Mandelbrot set collects all those z_0 shift values that if used with $z_1 = O$ as start value, will bring about a bounded sequence.
So: Mandelbrot $= \{ z_0 \ ; \ O = z_1 , z_0 = z_2 , z_3 , z_4 , \ldots$ are bounded $\}$

In spite of the Julia set being the simpler logically, the Mandelbrot set became more famous.
The two inter-relate delicately.

The mentioned five simple facts in the title give only a rough knowledge of the Mandelbrot set, but they are so easy to obtain, that form a definite start of the subject.
Yet these are never shown in the popularizing exposes.
To visualize the Mandelbrot set is best by regarding them as black points in the white background of all those points that cause divergence.

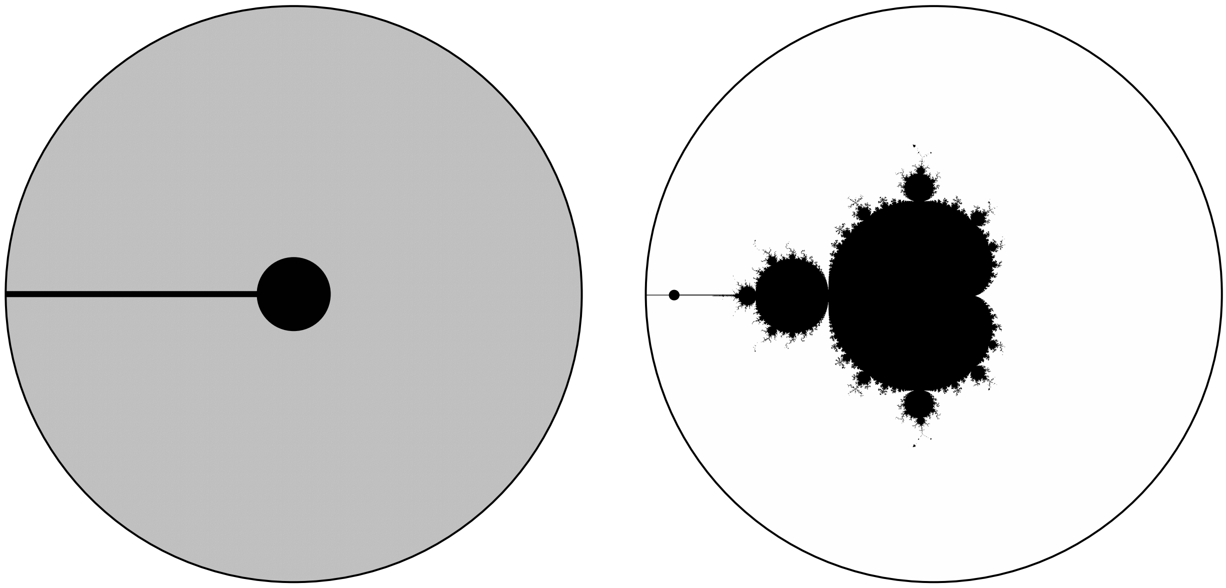
The first easy fact is that the Mandelbrot set is symmetrical to the x line.

The second fact says that if $| z_0 | > 2$ then we must have divergence.
So, the plane outside of the 2 radius circle is a definite "outer zone", it is white.

The third fact says quite oppositely that if $| z_0 | \leq \frac{1}{4}$ then we must have boundedness.

So, the disc of the $\frac{1}{4}$ radius circle is a definite "inner zone", it is black.
These are the trivial points of the Mandelbrot set.
Between these two circles lies a ring that is not obviously white nor black.
So, it is the ring where all the non trivial points of the Mandelbrot set must be.
Luckily, on the x axis, the points of the ring are trivial too. Namely, on the right of the trivial $\frac{1}{4}$ disc all points cause divergence, so are white, while to the left, all points cause boundedness, so are black and belong to the Mandelbrot set.

This will be the left picture, and the much wilder reality of the ring is on the right:



Now we'll come to proving the five facts.

1. Symmetry
Indeed, firstly, angles measured downward from x count as minus and so their doubling will be doubled downward too. Secondly, the shift is collected and it keeps symmetry, that is if a z_0' is symmetrical with z_0 then adding it to symmetrical z , z' will stay symmetrical.

2. Monotony of the outer zone
If $2 < | z_0 | \leq | z |$ then $| z^2 + z_0 | > | z |$.

Indeed, firstly $| z^2 + z_0 | \geq | z |^2 - | z_0 | \geq | z |^2 - | z |$.

But also, $| z | > 2 \rightarrow | z | - 1 > 1 \rightarrow (| z | - 1)^2 > 1 \rightarrow (| z | - 1)^2 - 1 > 0$

$\rightarrow | z |^2 - 2 | z | > 0 \rightarrow | z |^2 - | z | > z.$

Since $z_2 = z_0$ thus using z_2 as z we get that $| z_3 | > | z_2 |$.

Then using z_3 as z we get that $| z_4 | > | z_3 |$ and so on.

3. Boundedness in the inner zone
If $| z_0 | \leq \frac{1}{4}$ then the sequence cannot diverge.

Namely, if $| z | < \frac{1}{2}$ then $| z^2 + z_0 | < \frac{1}{2}$ too.

Indeed, $| z^2 + z_0 | \leq | z |^2 + | z_0 | < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

So, since $z_2 = z_0$ thus $| z_2 | \leq \frac{1}{4} < \frac{1}{2}$ indeed and so, $| z_3 |, | z_4 |, \ldots < \frac{1}{2}$ too.

4. Monotony on x
If $x_0 > \frac{1}{4}$ then $x < | x^2 + x_0 |$

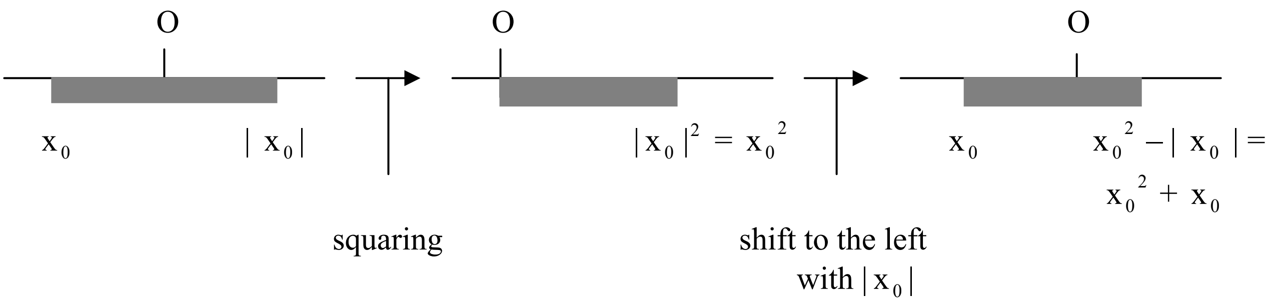
Indeed, firstly $| x^2 + x_0 | = x^2 + x_0$.

But also, $x_0 > \frac{1}{4} \rightarrow x_0 - \frac{1}{4} > 0 \rightarrow (x - \frac{1}{2})^2 + x_0 - \frac{1}{4} > 0$

$\rightarrow x^2 - x + x_0 > 0 \rightarrow x^2 + x_0 > x$.

So, actually $x_1 < x_2 < x_3 < \dots$

5. Boundedness from O to -2 on the x axis.
If x_0 is in $[-2, O]$ and x is in $[x_0, |x_0|]$ then $x^2 + x_0$ is there too.
Indeed, the $[x_0, |x_0|]$ interval is squared and shifted left to x_0:



Here $x_0^2 + x_0 \le |x_0|$ if $|x_0| < 2$ because:

$$0 \le |x_0| \le 2 \;\rightarrow\; ||x_0| - 1| \le 1 \;\rightarrow\; -1 \le |x_0| - 1 \le 1 \;\rightarrow\; (|x_0| - 1)^2 \le 1$$

$$\rightarrow\; x_0^2 - 2|x_0| = x_0^2 + 2x_0 \le 0 \;\rightarrow\; x_0^2 + x_0 \le |x_0|$$

A final exactification is needed for the above proofs in 2. and 4.
We indeed proved the monotony but, this does not instantly prove divergence. Indeed, it could be possible that the absolute values are always increasing, but never go beyond a certain value. This however would also mean that they approach a v value. This also implies that the actual points also approach a z point. The transformed of this z would have to be the limit of the limits and thus z itself, contradicting monotony.