The Well Ordering Theorem

Let f be an arbitrary function and S an arbitrary set.

The f widening of S is the S^f set defined as S itself if f is not defined on S and being $S \cup \{ f(S) \} = S + f(S)$ if f is defined on S.

If $S^f = S$ then we call S an f terminating set. This can mean two things by our definition, namely either that f is not defined on S or that $f(S) \in S$.

The set union of a B collection of S sets is $\bigcup B = \{ e \,;\, \exists S\,(e \in S \in B)\}$ which is the set that contains all those e elements that are elements in any S elements of B and thus is the combining of all elements of B. The $\exists S$ logical quantor means “there is such S that”.

The followings are a subjective description of the (s, f)-stages and (s, f)-widenings:
The s arbitrary set is regarded as a starting element.
The starting or first stage is $\{s\}$. The second stage is $\{s\}^f = \{s\} + f\{s\} = \{ s , f\{s\}\}$ if it is different from $\{s\}$ that is if $\{s\}$ was not terminating, that is if $f\{s\} \neq s$.
The third stage is $(\{s\}^f)^f = \{ s , f\{s\} , f\{ s , f\{s\}\}\}$ if it is different from the second. That is if $f\{ s , f\{s\} \} \neq s$ nor $f\{s\}$.
Similarly we define the wider fourth and so on stages.
The first infinite stage comes about by a union widening of all the earlier stages, that is as:
$\{ s , f\{s\} , f\{ s , f\{s\}\} , f\{ s , f\{s\} , f\{ s , f\{s\}\}\} , \ldots \}$
This can again be f widened if it is not a terminating stage.
The stages up to any point are called the widenings.
So the first widening is $\{\{s\}\}$ containing only the $\{s\}$ stage.
The second widening is $\{ \{s\} , \{ s , f\{s\}\} \}$ containing the stages $\{s\}$ and $\{ s , f\{s\}\}$.
The first infinite widening is:

$$\{ \{s\} , \{ s , f\{s\}\} , \{ s , f\{s\} , f\{ s , f\{s\}\}\} , \ldots \}$$

The first infinite “closed” widening is:

$$\{ \{s\} , \{ s , f\{s\}\} , \{ s , f\{s\} , f\{ s , f\{s\}\}\} , \ldots , \{ s , f\{s\} , f\{ s , f\{s\}\} , \ldots \} \}$$

So we added the combined stages to the widening and thus again obtained a widest stage.
This closing or union widening is important because now the f widening can continue from this widest last element.
The S stages can not be grasped by rules directly because they melt into each other.
The W widenings luckily can be grasped and they at once give the stages as their elements.
So, the word “stage” will only mean being element of W.
The crucial concept in describing the widenings is the “beginning”. This is the stages up to any point and to avoid this imprecise concept of “up to any point”, simply requires that all elements in the beginning should be proper subsets of all the stages that are outside. So:

If $B \subseteq W$, B is non empty and $(S \in B , T \in W - B) \to S \subset T$ then B is called a beginning.

This makes the whole W a beginning too because then $W - B$ is empty with no T in it and thus the required implication is true for “all” T. If B is not W we call B a proper beginning.
If B has a widest element, that is $\bigcup B \in B$ then we call B closed, otherwise open.
This same distinction is used for W itself. The $W - B$ set is called the end of B.
The proposed precise description of the (s, f)-widenings is this:

1. $\{s\} \in W$

2. $S \in W$ and $S \neq \bigcup W \rightarrow S^f \neq S$ and $S^f \in W$

3. B is open proper beginning $\rightarrow \bigcup B \in W$

4. $S, T \in W \rightarrow T \subset S$ or $S \subseteq T$

A special consequence of 4. is that $\{s\}$ is subset of all stages.
Indeed, $\{s\}$ has only itself as subset. Thus of course s is element of all stages.
An other third way of saying this is that $\{\{s\}\}$ is a beginning.

If B is a closed beginning then $\bigcup B \in B$ and this is the widest stage in B.
If B is also proper then $\bigcup B \neq \bigcup W$ so by 2. $(\bigcup B)^f$ is wider than $\bigcup B$ so $\in W - B$.
We claim that all other T stages in the $W - B$ end are wider.
By 4. it's enough to show that $T \subset (\bigcup B)^f$ is impossible. But $\bigcup B \subset T$ and so T would be in-between $\bigcup B$ and $(\bigcup B)^f$ which is impossible because $(\bigcup B)^f$ has only one extra element.
This implies that if B is a closed proper beginning then $B + (\bigcup B)^f$ is again beginning.

If B is an open proper beginning then by 3. $\bigcup B \in W - B$.
We claim that all other T stages in the $W - B$ end are wider.
By 4. it's enough to show that $T \subset \bigcup B$ is impossible.
Every $e \in \bigcup B$ is coming from an S with $S \subset T$ so $e \in T$ too and thus $\bigcup B \subseteq T$.
This implies that if B is an open proper beginning then $B + \bigcup B$ is again a beginning.

These two results about widening the beginnings help to prove a crucial result:

I. For any two W, W' widenings, one is a beginning of the other.

We combine those beginnings of W and W' that are definitely common.
This total B set will be beginning in both sets. Indeed:
B is non empty because $\{\{s\}\}$ is a common beginning. Also, for every S in B and T outside in either W or W' we have $S \subset T$ because S was in a beginning and T outside of it.
So this B is also the widest possible common beginning.
If B is not proper in one of them that is the full W or W' then of course we are finished.
But this has to be the case because if B were proper in both then we could add $(\bigcup B)^f$ if B is closed or $\bigcup B$ if B is open and we would get a wider common beginning by the aboves.

We are finished but observe that this same argument can be used for two (s, f) and (t, g) widenings too. Here of course the two W and W' can not be actually parts of each other but one will be similar to a beginning of the other. By this we mean that the elements can be assigned one to one so that the new f and g values are also assigned. The s and t starters are assigned to each other as start since they are not function values yet.

A kind of reverse of I. is this:
II. The beginnings in any W are widenings themselves.
This can be verified by simply checking the rules for a beginning.

In a closed widening the $\bigcup W$ widest stage can be a terminating stage and if it is, then we call the W widening a terminating one too. Our intuitive feeling is that such terminating widening exists by simply "let" the f and union widenings "grow" as far as possible.

Rule 2. allows termination only for $\bigcup W$ and this gives a good idea to actually prove that there is terminating widening. Namely, to combine all possible widenings.
A small nuance is that such combining might lack a widest stage and so actually our definition of a terminating widening should be:
$D = \bigcup Y + \bigcup \bigcup Y$ with $Y = \{ W ;$ 1. and 2. and 3. and 4. $\}$
If this D is indeed a widening, so obeys the four rules, then it's almost trivial that it has to be terminating. Indeed, it has to be the widest possible widening and then if it were not terminating that is if $(\bigcup D)^f = \bigcup D$ were then $D + (\bigcup D)^f$ were a wider widening.

A trivial consequence of I. is that only a single terminating W can be.
So, what we just proved means that above where we introduced D as "a" terminating widening, we could have said "the".
The only thing remains to be proven is that the four rules inherit to D.

1. is trivial because $\{s\}$ remains in the $\bigcup Y$ combined set.
2. could be said also as: $S , T \in W$ and $S \subset T \rightarrow S^f \neq S$ and $S^f \in W$.
So both 2. and 4. are trivial because any two S , T in D came from W , W' and by I. one contains the other as beginning so S , T both were in that wider one obeying 3. and 4.

3. requires both I. and II. Let B be an open proper beginning in D.
Thus there is some T stage of D not in B and it came from a W.
The $S \subseteq T$ stages in D came from all kinds of widenings but in each the $\{ S' ; S' \subseteq S \}$ sets are closed beginnings by 4. so they are also widenings by II. and so beginnings of W by I.
Thus the B beginning of D is already a beginning in W and $\bigcup B$ was already in W.

A well ordering of an F set is any W widening so that $\bigcup W = F$.

Well Ordering Theorem : Every F set has well ordering.
Proof: We have to find an s and f so that for some W (s , f)-widening $\bigcup W = F$.
Since we only know anything special about D namely that it is terminating, it would be logical to find s and f so that we must have $\bigcup D = F$. Amazingly this is quite easy.
We only have to assume that there is a c subset choice function defined on all real that is non empty subsets of F so that c is picking an arbitrary element from each S subset.
In short $c(S) \in S$.
This c will provide our s starting element as c (F) and then $f(S) = c(F - S)$ for all $S \subset F$.
Clearly with this f if an (s , f)-widening is terminating then termination can only happen at F.
So indeed our D will do.

Comparability Theorem : For every two F , G sets one is equivalent to a subset of the other.
Proof: We have to well order them and then by our earlier mentioned generalization of I. one of the sets will be similar to a beginning of the other. This of course means equivalence too.

A comparing partial equivalence between F and G so that one is the full set can be obtained directly as a terminating widening without well ordering these sets!!!!!
Indeed, we can simply regard a c choice function for all sets of pairs from F and G.
Then the A set of all possible pairs is such too and c (A) gives (s , t).
For every E set of pairs that is not full in neither F nor G we then can define a $h(E) = c(E^*)$ with E^* being those pairs that have no element occurring in E.
This (s , t) starting element and h function can only terminate at an equivalence being full in either F or G.

The trivial generalization and thus simplification of rule 3. could be to simply omit the open condition about B. Indeed, for closed ones $\bigcup B \in B$ and so $\in W$ too anyway.

A much more interesting generalization is to omit the beginning condition at all.
Of course, beginnings included the condition of being non empty and this is still needed, so actually our new rule would be: $0 \neq C \subseteq W \rightarrow \bigcup C \in W$.
You might think that I used here the letter C for collection or combining but actually it is for a new concept "cofinal" which goes to the heart of why this generalization is true.
The simple fact is that for an arbitrary C the $\bigcup C$ will collect exactly the same elements as $\bigcup B$ if C "goes all the way", that is cofinal with B. Meaning that neither C has element that would be wider than all B elements and vice versa B has no element that would be wider than all C elements. Indeed, then first of all C is subset of B and C may skip some S stages in B but C contains wider than S stages and so will eventually collect all the elements of S.
I'll come back to cofinals but first show a totally opposite direction of generalizations.
I started through the intuitive duality of the alternating closed and open beginnings and used the corresponding two f and union widenings. But this also implied that the W– B end section always has a first stage namely $\bigcup B$ if B is open or $(\bigcup B)^f$ if B is closed.
These combined simply says that: All ends have a first stage = E is end of W $\rightarrow \bigcap E \in E$.
Here this set intersection is defined as $\bigcap E = \{ e ; \forall S (S \in E \rightarrow e \in S) \}$.
The $\forall S$ logical quantor means "for all S" and thus $\bigcap E$ collects those e elements that are in every S stage in E, so indeed it is an intersection or common part of all E members.
Of course, here with S stages, due to 4. they form a chain and so this intersection is actually a first stage. The main difference from the beginning oriented 3. rule is that now this first stage of E has to be in E not merely in W as the union widening was.
This end oriented view again can be generalized from ends to any Q subsets of W as:
A. $0 \neq Q \subseteq W \rightarrow \bigcap Q \in Q$.
The use of the letter Q has again an importance, namely to introduce a heuristic P-Q method.
Here P stand for any P (S) stage property for which we know that:

a. $P(\{s\})$
b. $P(S) \rightarrow P(S^f)$
c. $(P(S)$ for all $S \in B) \rightarrow P(\bigcup B)$

In this case P (S) must be true for all $S \in W$.
Intuitively this is evident because P inherits to all stages, but the exact argument is this:
Suppose there were some T stages that were not P kind and let their set be Q.
Thus Q is non empty and so by the above claim $\bigcap Q \in Q$ were a first such.
But $\bigcap Q$ can only be either $\{s\}$ or S^f or $\bigcup B$ with these last two being earlier and thus S being P kind and B only containing P kind of stages. So then a. , b. , c. would imply the same for $\bigcap Q$, contradicting that is was not P kind.
So, we needed this intuitively obvious claim that:
B. All stages can only be three options, $\{s\}$ or S^f or $\bigcup B$.

This indeed can be easily proved from our results but amazingly A. and B. define the widenings already. This can be seen at once by the Q-P method because all rules are P (S) properties so that: $\{s\}$ obeys them and they inherit to f and union widenings.