Contents

1. Introduction

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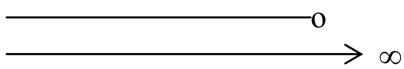
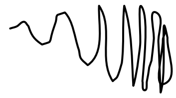
Cantor was trying to solve some problems in Analysis and that’s how he stumbled upon the fundamental concept of equivalence between sets in general. In short, Set Theory grew out of Analysis. But soon it became obvious that Analysis itself can be made much more exact if we regard it as a field of point sets. Points of course belonged to the old field of Geometry, but there the lines and circles were regarded as separate basic objects, not as sets of points. Hilbert re-axiomatized Geometry with the new set meanings, but basically followed the old Euclidian axioms. Indeed, just because we regard the line as a set of points it doesn’t change anything because then, instead of the line, the straightness becomes a basic undefinable concept. Still, the set theoretical view helped to improve the Euclidian system in the small details.
After Set Theory, the rewriting of Analysis has started, but actually it never finished. It turned out that if we start with points instead of numbers, then quite new things become important, so the whole field strays away from the old Analysis. This new field is topology.
The fundamental concept both in Analysis and in this new field is the continuous function. Function meant assigning numbers to numbers, while now assigning points to points. The set theoretically exact form of functions as sets of ordered pairs is unnecessary and immaterial. Rather, it is the abstraction of continuity and the combining of this continuity with the set theoretically so fundamental one to one function, that is equivalence, that brought in the new meanings. While in Set Theory, equivalence is the tool that can compare the size of sets, here in topology it is an added extra feature beyond continuity.
Continuity in its naïve meaning should refer to sets themselves. A line, a curve, is continuous if we can follow the pencil through them. This is of course very imprecise, because uses the physical motion, which involves time itself. The ancient paradoxes of sophist greek philosophers, questioned the reality of such continuity. Indeed, how can we go from point to point, when in between there are infinite many other points. The newest level of attack on our intuitions about continuity are the now so fashionable, fractals. The fundamental truth is that mathematics is using a trick to define continuity. Instead of defining it for sets, it accepts that the geometrical concept of a distance or line is continuous and then, transfers this to other sets by defining the continuity of the transferring function. This trick was already used in analysis, and it remained the same in topology. The continuity of a distance itself is simply not asked anymore, rather assumed as obvious.
Continuity of functions is the only exactly defined mathematical concept, but it still has a naïve meaning too. Namely, a function is continuous if it assigns to “nearby” points, also “nearby” ones. In a more detailed, but still naïve way: for arbitrary close points to a fixed P, the function values should be also close to $f(P)$.
The heuristic exact definition of this is achieved by the proper use of the logical quantors “every” = $\forall$ and “there is” = $\exists$. It turns out that the first quantized variable is the distance from $f(P)$. Indeed, instead of emphasizing the arbitrary closeness, it is sufficient to require that for every ε surrounding of $f(P)$, there is a δ surrounding of P, so that within this, the f function will remain in the ε surrounding of $f(P)$.
An other way to define continuity is through the fundamental concept of approach. An S set is approaching a P point, if there are arbitrary close elements of S to P. Or again avoiding the subjective concept of “closeness” and rather just emphasize the important “every” feature, we get the exact definition that:
In every surrounding of P, there is an element of S besides P, if it is an element of S itself.
Then, continuity means that the function keeps the approaches, not only for the whole domain, but for any subset. In other words, if an S approached a P in the domain, then $f(S)$ should approach $f(P)$ in the range too. As it turns out, this version of continuity is identical with the ε, δ definition.

Since continuity is defined only by the inside properties of the domain, it is surprising that it can guarantee the transfer of a feature that depends not only on the sets but their complementer surrounding too. This feature is "closedness". A set is called closed if all the approached elements by it are already elements. Or in other words, the set doesn't approach any outside points. So, the fundamental fact is that if f is continuous and S is a closed set in the domain, and is bounded, in other words is inside a finite box, then f (S) is again closed and bounded. We'll prove this exactly, but the basic idea why it is true can be already explained here:
Indeed, suppose f (S) were approaching a point outside of f (S). Lets choose a sequence of points from f (S) that is also approaching this outside point and only approaching that one. Then, we can find a sequence in S, so that the f values of this sequence is the chosen one in the range. But S was closed, so this sequence can only approach inside points! And indeed, it must approach some points, because it is bounded. If one of these is P, then by continuity, the sequence that we choose in the domain can only approach f (P) in the range.
A fundamental "weakness" of continuity comes if the domain is not closed, that is, it approaches points outside. Then the function can distort the distances "infinitely". And here the "infinitely" is meant both locally and globally. Indeed, an interval open in one end or an infinite half line, can be changed into an infinitely wiggling line:





These "weird" functions were not only continuous, but one to one.
As we mentioned, just as the one to one function or equivalence is the basic concept of Set Theory, this combined with continuity is the basic concept at point sets.
The reasons are totally different though. Equivalence was the way to compare infinites. Here the different infinities are immaterial and instead the different shapes are important. But we still didn't tell what we mean by combining the equivalence with continuity and also we saw above that one to one continuous function can distort shapes. In a sense, both of these have a common "solution".
Lets remember that sets without disappearing points are called closed. Bounded and closed sets are called compact. One to one functions that are continuous in both direction are called homeomorphisms. Then:

a.) Homeomorph picture of a compact set is again compact.
b.) One to one continuous function on a compact set is continuous backwards too, so is a homeomorphism.

This at once shows that compact sets are "designed for" the basic concept of homeomorphism.
Compact sets are very different! A single point, a closed interval or curve, a circle, square or a curved loop, a ball with its surface sphere, just a surface sphere, these are all compact sets. We pretty much feel intuitively, which compact sets can be homeomorph and which can not. For example, a circle, square or any curved loop are homeomorph with each other but they can't be homeomorph with an interval or curve with two ends. Indeed, the ends can not go into a middle point! What we have to realize here is the big new idea that was neglected in Analysis, namely that local properties can lead to global. Continuity and "one to one"-ness are local but they together define the global features of being a loop or not. So we can distinguish shapes, regardless of their particular exact point set representations. After we shout, "eureka", by realizing that we just entered Topology as opposed to Analysis, we have to emphasize two things. Firstly, by entering Topology, we also entered a whole new wonderland where the most obvious facts can be the hardest to prove. Even more disturbingly, the proofs will be going along totally different lines as our intuitions. A fundamental example is the following:
Just as a circle can not be homeomorph to an interval with its two ends, a sphere can not be homeomorphic to a disc with its perimeter. Indeed, the perimeter points of the disc couldn't go into the sphere, where every point is "inside". The homeomorphisms of compact sets will obviously keep the approached points, so the only crucial part is to prove that inner points can't become mixing elements and vice versa. Or in short, that inner points remain inner.

An inner point has a little ball around it, fully inside the set and we feel that such ball even if it is distorted by the homeomorphism into a different shape, it will still have on its surface the image of the surrounding sphere. So the center can't go into the surface and thus is inside, having again a ball around it, fully inside too. But here we came back to where we started from, that is the homeomorphism of spheres. A set that only has inner points is called open, and so the invariance of the inner points can also be said as the invariance of openness. The names already suggest the fundamental fact, that open and closed sets form complementer pairs. This lead to a totally new abstract extension of Topology, which is called General or Set Topology. Strangely, (and stupidly) the invariance of openness was called originally the invariance of domain, but even more strangely when Hilbert listed his famous unsolved problems in 1900, he didn't mention this. Alexander solved it much later with methods that combine Geometry and Algebra. Later, simplified proofs were found and we'll show such, but it still relies on coordinate arguments. Even though our intuition is direct about the invariance of the inner and mixing points, the proof relies on another invariance of compact sets. Namely, whether they separate the space or not. A sphere is obviously separating it because we can't go from the inside out, without crossing the sphere itself. The homeomorph picture of a sphere is a shapeless "bag", but it still will separate the space. Originally, Jordan proved this in two dimension, that is that in the plane, any homeomorph picture of a circle will separate the plane. Here the circle can be traveled around and then this will define a similar full round, but with possible back and forth turnings. In three dimension, the situation is much harder because there is no fix circling direction.

The second fact that we have to mention about Topology versus Analysis is that the topological view was ignored in classical Analysis, exactly due to the above mentioned hard reasonings.

Indeed, Formalism was in full gear already and desperately tried to avoid visual meaning in favor of rigorous derivability. Best example is the Fundamental Theorem of Algebra. Gauss gave half a dozen different proofs for it, but none of them mentioned the simple topological reason behind it. A polynom is obviously a continuous but not one to one function in the complex plane. This means that traveling around a circle the assigned points might go around more than once before they return and worst of all their path can cross itself several times. So the Jordan inside outside argument can not be used but a different more complicated continuity can be used: If we increase the traveled circle, then the self crossing multi loop image also increases because in a polynom the highest power will dominate all the others. Thus, if we regard the full domain of the plane as this increasing concentric disc then it's "obvious" that sooner or later the range takes up all points of the plane too. This includes 0, so every polynom has a root. Gauss avoided all this because this "obvious" fact is quite hard to prove.

The most important fact is a third one though:

Just as the equivalence is the fundamental concept of Set Theory, but is unsuccessful in perfectly distinguishing the set sizes, due to the unsolvability of the Continuum Hypothesis, the homeomorphism is the fundamental concept of Topology, but is unsuccessful in perfectly distinguishing the different shapes. But here this unsuccessfulness is two fold.

We said that the magic of homeomorphism is that such totally local description leads to distinguishing global shapes. A circle or a loop can't be homeomorph to a two ended curve like a shoelace. If we make a loose knot on a shoelace, not touching itself, then this has no effect on the homeomorphism of it, which is logical because we can continually move the shoelace into the knotted position. But, if we make a knot and then glue together the ends of the shoelace, then we get a knotted loop, that can no way be moved into an unknotted loop. The knotted and unknotted circles are homeomorphic, but they are "different". Similarly, two rings as a set are homeomorphic to two linked rings. So homeomorphism doesn't describe knots, links and other entanglements of sets. This is the more natural "defect" of homeomorphism. To find a stricter homeomorphism that distinguishes these entanglements of the shapes too, we can introduce a whole continuous set of homeomorphisms from a 0 time to a 1 time, so that to every moment of the [0 , 1] interval we'll have one. Then indeed, we can't get from an entangled version to a non entangled. But this continuous deformation or motion was working mainly because it was embedded into the whole surrounding space.

So the question arises whether we can avoid the use of time completely and merely require a homeomorphism of the whole space, that “moves” one set into an other homeomorph one. For certain sets, this will give the same distinguishing of entanglements. This at once shows that complementers, that is the sets remaining of the space taking out a set from it are vital. Then, the weirdnesses of homeomorphisms can be shown by how different non homeomorphic complementer sets can appear for homeomorphic sets.

Finally, we might think that with the birth of Topology and thus replacing Analysis on a new, more visual foundation, Formalism was avoided. But, just the opposite happened. In General, or Set Topology, the abstract definitions are never explained in terms of the real space concepts that they try to imitate. In the more down to earth, so called Geometrical Topology, the intentions are much better, but the end results are still chaotic and badly explained text books. In 1932, Paul Alexandroff wrote a little booklet called The Elementary Concepts of Topology, with a preface by David Hilbert. This book was written with the best intentions, but it failed to even outline the difference between Topology and Analysis and between Set Topology and Geometrical Topology. Another example is R.H. Bing’s writings. His article in the American Mathematical Monthly called Challenging Conjectures is an honest and simple approach, but it is not followed with text books in the same vain. Instead, his Geometrical Topology leaves all the important points in darkness and proceeds with perfect Formalism.

2. Convergence and Continuity

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If a set “approaches” only one point and the set is also bounded, that is contained in a sphere or interval cube and thus can’t have points “going away” to infinity, then all the points are “around” the approached point. This situation can be grasped by two different definitions too. If the approached single P point is known, then we can tell that most of the points of the set, namely infinite many of them are inside any surrounding of this point, while outside there are only finite. This is what we’ll call convergence to P. If however, we don’t know where the single approached point is exactly, only that it is inside an interval or surrounding, then we can just list all the finite many points outside, then take a smaller surrounding inside and continue listing the new outside points again, and so on. This way, we’ll get a sequence of points, so that if we leave out more and more of these sequence points from the set, then it will be in smaller and smaller surroundings. So we’ll say that the set is diminishing by such sequence. Of course, this will actually locate the single approached point too. The identity of these three kinds of definitions, that is of single approached point, convergence and diminishing is what we prove first in the followings. If instead of the S set, we start with a sequence of points, then the same point can be repeated many times. In fact the same single point can be repeated only after a while. This should be regarded as a limit of the sequence, even though there are only finite many actual points appearing in it. So the single approached point is not a good definition for sequences. Luckily, the other two, the convergence and diminishing can be easily modified for sequences. Even better is that these two will mean the same thing again. This will be used to prove the basic properties of continuous functions.
In the followings, we define exactly these concepts:

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1.) $S < \varepsilon$ if any two points of S are closer than ε.
S is bounded if $S < \varepsilon$ with any ε.
$S_1, S_2, \ldots \rightarrow 0$ or $S_n \rightarrow 0$ if for every ε there is an N so that:
If $n > N$ then $S_n < \varepsilon$.

2.) S is approaching a P point if in any surrounding of P, there is point of S other than P. Then in any surrounding of P, there must be infinite many points of S, because we can choose smaller and smaller surroundings.

The unapproached elements of S, have a surrounding in which they are the only S points. Thus, these elements are called the isolated points of S.
The unapproached points that are not element of S, have a surrounding where there are no S points at all. Thus, these points are called outer points of S.

The approached points are not as easy to classify. The best to first regard those points that are not only approached by S, but surrounded completely by S points. In other words, such P have a surrounding where all other points are S points. If P is also element of S, then such P has a surrounding where all points are S points. Thus, these points could be called inner points. If P is not element of S, then it could be called a hole.

Those approached P points that are not surrounded completely by S points should be called mixing points, because they have closer and closer, both S and outside points.
Then these again fall into two cases depending on whether P itself was in S or outside.
Thus, we have mixing elements and mixing non elements.
The following shows the achieved classification of points:

unapproached point
- element = <u>isolated point</u>
- non element = <u>outer point</u>

approached point
- fully surrounded by S
  - element = <u>inner point</u>
  - non element = <u>hole</u>
- non fully surrounded by S = mixing point
  - = <u>mixing element</u>
  - = <u>mixing non element</u>

A special name for combined type:
approached non element = hole or mixing non element = disappearing point

Every point is exactly one of the six final possibilities. The approached non element could be called disappearing points of S, because S approaches these, but "then" it "disappears".

3.) An S set converges to a P point, in short, S → P or lim S = P if:
S is infinite but, outside any surrounding of P there are only finite many points of S.
S is convergent if it converges to a point.

4.) S diminishes by a $P_1, P_2, \ldots$ sequence of its points, if S is infinite and, removing $P_1, \ldots, P_n$ from S, it becomes arbitrary small. That is, $S - \{P_1, \ldots, P_n\} \to 0$.
S is diminishing if it diminishes by a sequence.

5.) A $P_1, P_2, \ldots$ sequence converges to a P point, in short: $P_n \to P$ or $\lim P_n = P$ if:
outside any surrounding of P, there are only finite many members.
In other words, from a member, all are inside the surrounding.
A sequence is convergent if it converges to a point.

6.) $P_1, P_2, \ldots$ is diminishing, if $\{P_{n+1}, P_{n+2}, \ldots\} \to 0$.

T

1.) Bolzano Weierstrass: Bounded infinite set approaches some point.

2.) $S \to P$ ↔ S is bounded and approaches P but no other point.

3.) S converges to a point ↔ S diminishes by a subsequence.

4.) $P_n \to P$ if and only if:
Either $\{P_1, P_2, \ldots\}$ is finite and $P_1, P_2, \ldots$ stays P from a member.
Or $\{P_1, P_2, \ldots\}$ is infinite, converges to P and no point appears infinite many times in $P_1, P_2, \ldots$ except maybe P.

5.) $P_1, P_2, \ldots$ is diminishing if and only if:
Either $\{P_1, P_2, \ldots\}$ is finite and $P_1, P_2, \ldots$ stays P from a member.
Or $\{P_1, P_2, \ldots\}$ is infinite, diminishes by $P_1, P_2, \ldots$ and no point appears infinite many times in $P_1, P_2, \ldots$ except maybe the P, determined by the diminishing of $\{P_1, P_2, \ldots\}$ according to 3.).

6.) Cauchy: $P_1, P_2, \ldots$ is convergent ↔ $P_1, P_2, \ldots$ is diminishing.

P

1.) Let I contain S! One of the halved subintervals of I contains again infinite many points of S. Choosing this and repeating the halvings, we narrow down to an approached point.

2.) If S $\rightarrow$ P, then it approaches P, because in any surrounding of P, we have infinite many points of S. Also, it can't approach any other point Q, because taking a surrounding of P that doesn't contain Q, there are only finite many points outside.
In reverse, if S is bounded and approaches a single P, then outside any surrounding of P, it can have only finite many points. Otherwise by 1.), the outside points as a set would approach an other point, contradicting that there was only one.

3.) If S $\rightarrow$ P, then S diminishes by any $P_1, P_2, \ldots$ which are the outside points of a sequence of smaller and smaller surroundings around P.
If S diminishes by a $P_1, P_2, \ldots$ sequence of it, then first of all S is bounded because $S - \{P_1, \ldots, P_n\} < \varepsilon$ and $\{P_1, \ldots, P_n\}$ is bounded too, because it's finite.
So by 1.), S approaches a P and it can't approach an other Q. Indeed, otherwise leaving out finite many points from S, there would still remain arbitrary close points to P and Q, so $S - \{P_1, \ldots, P_n\}$ couldn't be in arbitrary small.

4.) Both the finite and infinite cases are trivial.

5.) The finite case is trivial. At the infinite, the diminishing of the sequence is not the same as its points as a set, because in the first $\{P_{n+1}, P_{n+2}, \ldots\} \rightarrow 0$, while in the second, $\{P_1, P_2, \ldots\} - \{P_1, \ldots, P_n\} \rightarrow 0$. These two sets are not the same because a point of $\{P_1, \ldots, P_n\}$ can appear among $\{P_{n+1}, P_{n+2}, \ldots\}$, but it is taken out in $\{P_1, P_2, \ldots\} - \{P_1, \ldots, P_n\}$. In spite of this, if every point appears only finite many times in the sequence, then $\rightarrow 0$ is the same for the two sets. If a P point appears infinite many times, then $\rightarrow 0$ in both cases implies that P must be the limit.

6.) Follows from 4.) 5.) and 3.) applied to $\{P_1, P_2, \ldots\}$

D

An f function of points is continuous <u>at</u> P if P is approached point of D (f) and:
for every $P_1, P_2, \ldots \in D(f)$, if $P_n \rightarrow P$ then $f(P_n)$ is convergent.

T

If f is continuous at P, then:
For all $P_1, P_2, \ldots \in D(f)$ and $P_n \rightarrow P$, the $\lim f(P_n)$ points are the same.

P

Suppose (P_n) and (Q_n) in D (f) both converge to P, but $\lim f(P_n) \neq \lim f(Q_n)$.
$P_1, Q_1, P_2, Q_2, \ldots \rightarrow P$ too, but $f(P_1), f(Q_1), \ldots$ can't be convergent, by Cauchy.
Indeed, $\overline{f(P_n)\, f(Q_n)} > 0$ due to the different limits.

D

This common $\lim f(P_n)$ point is denoted as $\lim\limits_{P} f$.

D

f is continuous <u>in</u> P if it is continuous at P and $P \in D(f)$.

T

If f is continuous in P, then $\lim\limits_{P} f = f(P)$:

P

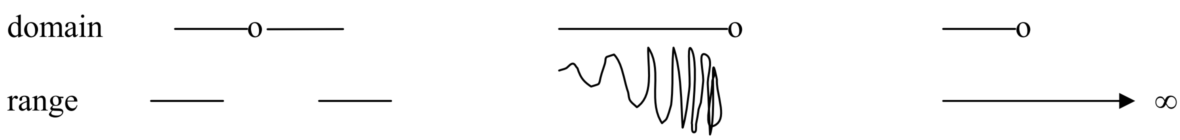
Suppose (P_n) in D (f) converges to P, but $\lim f(P_n) \neq f(P)$.
$P_1, P, P_2, P, \ldots \rightarrow P$ too, but $f(P_1), f(P), \ldots$ can't be convergent, by Cauchy.

D 1.) f is continuous, or a map if it is continuous in every $P \in D(f)$.

2.) f is extendedly continuous, or an extended map if it is continuous, D (f) has disappearing points, and f is continuous at these points too.

R Non extendable maps:

“tearing apart” “dislocation” “taking to infinity”



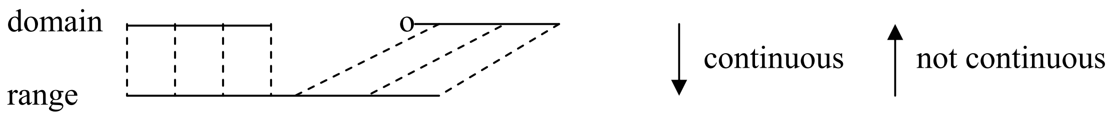
These examples become even more interesting in two or three dimension.
The tearing apart will be the “stretching” of a single missing point into a disc or ball. So, not only the sequences on the left and right will be torn apart, but all around. Then in then range they all will approach different points of circle’s perimeter or ball’s surface.
The taking to infinity can similarly happen in all directions.
The dislocation is much harder to visualize, but the weird fact is, that:
If the center point is removed of a disc or ball, then it can be continuously wrinkled, so that arbitrary close points to the center will all be further than a fixed distance.

D A set is closed if all the points that it approaches are elements of it. In other words, if the set has no disappearing points. Bounded closed sets are also called compact.
The f function symbol can be used not only for the $P \in D(f)$ as $f(P)$, but similarly for any $S \subseteq D(f)$, as $f(S)$ meaning the set of all $f(P)$ for $P \in S$. This is also called the picture of S.

T Compact S set’s continuous picture is compact too.

P Suppose $f(S)$ were not a.) bounded or b.) closed. Then we could find a sequence in it, that a.) goes to infinity or b.) converges to a disappearing point.
So there were $f(P_1), f(P_2), \ldots$ points in $f(S)$, so that only finite many of these were:
a.) inside of any I interval, or b.) outside any surrounding of an $R \notin f(S)$.
By boundedness and closedness of S using Bolzano Weierstrass:
$P_1, P_2, \ldots$ would have a $P^1, P^2, \ldots \to P \in S$ subsequence.
Then by continuity: $f(P^1), f(P^2), \ldots \to f(P)$.
But this would be impossible if only finite many of $f(P^1), f(P^2), \ldots$ were in any I or outside a surrounding of R not containing $f(P)$.

R In the aboves, we always visualized the map to be one to one, but we can have common value points. Indeed, the name reflects this because maps are three dimensional objects projected into two dimension.
One to one functions of points could be called displacements and if they are continuous, then we can visualize them as flexible but still physical displacements of whole sets. Strangely, if a displacement is continuous in one direction, then its inverse still doesn’t have to be continuous, because we can create approaches:



Again, we can even bring in a point from “infinity” to the middle of the range. So to avoid all this we need compactness.

An other plausible way to imagine a continuous displacement is as a one to one map which could be called a modeling of the sets. We'll stick with this visualization. Since the word model is used already in Set Theory, we'll use the accepted name, homeomorphism:

D If a one to one function is continuous in both directions, then it is called a homeomorphism and the $f(S)$ pictures as homeomorph pictures.

T If f is a one to one map and $D(f)$ is compact, then f is homeomorphism.

P All we have to show is that the reverse of f is continuous too.
The proof resembles the previous one.
Suppose for an $f(P_1), f(P_2), \ldots \to f(P)$, the $P_1, P_2, \ldots \to P$ wouldn't stand.
By boundedness and closedness of $D(f)$ using Bolzano Weierstrass:
$P_1, P_2, \ldots$ would have a $P^1, P^2, \ldots \to Q \in D(f)$, $Q \neq P$ subsequence.
Then by continuity: $f(P^1), f(P^2), \ldots \to f(Q) \neq f(P)$, contradicting what we supposed.

R The last two theorems reveal that compact sets play a special role in continuity.
We could go on and state more results for compact sets, but our examples of lost or not reversed continuity suggest that much more is true. In particular for a homeomorphism, we feel that not only the compact sets remain compact in both directions, but the three types of points in a compact set, the isolated, the inner and the mixing are also preserved. The isolated ones are trivial by the definition of continuity for only approached points. So, all this boils down to the inner points remaining inner. This coincides with an other abstract direction relying on the inner points.

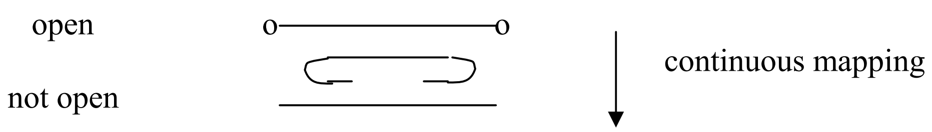
D A set is open if every element of it is an inner point.

R Earlier we introduced our own names from the local properties and agreed that a property used for a set, meant the existence of such point, while the "un-" prefix was used for the non existence of such points. This is very logical, and unisolated, undisappearing, unouter, uninner mean exactly what they say, that is sets without, isolated, disappearing, outer or inner points. Of course, the last two can be also said with better sounding interval based properties, as nowhere empty or nowhere full. The undisappearing sets are called closed but up until now we didn't encourage this name. Now, with the introduced open sets, we'll change this too, because open and closed sets form complementer pairs. The open sets themselves could not be called by a local property, because having only inner points is neither the existence or the non existence of a single type of points. Of course, the complement is closed, that is undisappearing so the open sets could be called as unapproached by their complements.

T Open set's complement is closed and closed set's complement is open.

P An S set disappears in P if and only if P is isolated point or mixing element of $\bar{S}$.
So not having any disappearing point for S is the same as $\bar{S}$ not having isolated or mixing elements, that is only having inner points as elements.

R Openness is not conserved by maps, because we can fold back sets, creating mixing points.



On the other hand, we feel that homeomorphism keeps the openness! This, in spite of its plausibility, is very hard to prove. In fact, this whole book is for this purpose!

When continuity, that is map, keeps some property we usually call it as conservation.
When homeomorphism keeps something, we'll call it invariance!
So the invariance of openness is a mystery because it's a very simple and very plausible fact and yet, it's very hard to prove. But this mystery is much deeper! The complementarity of openness and closedness suggests that there is a deeper symmetry. And indeed, it turned out that the whole subject of point sets and their maps and homeomorphisms can be handled abstractly! This direction is called Set Topology, to distinguish it from the topology of the normal space. The fundamental idea is the following: We can take any set as the "set of points" or the space and prescribe some subsets of this as the chosen "open" sets. Their complements will be the closed ones. The basic laws which we see for open sets in the real space are:

1.) Union of open sets is open. Thus, intersection of closed sets is closed.
2.) Union of finite many open sets is open. Thus, intersection of finite many closed is closed.
3.) The full set (space) is both open and closed. So the empty set is too, by complementarity.

The real trick that makes these abstract spaces useful is that the concept of the surrounding of a P can be replaced by any open set containing P. Then of course, we can make theorems by assuming special features of the topology, that is openness defined in the set and predicting things we know from our real space. Also, continuity and homeomorphism can be defined.
But now, the invariance of openness becomes the definition of homeomorphism itself, so the whole difficulty of proving it disappears. We have to realize that the big advantage of such Set Topology is that we can work without the concept of distance! If we make a big jump backwards and introduce an abstract distance instead of the openness, then we get the concept of metric spaces. These can easily be turned into topological and then the question of invariance of openness can be raised again because surroundings have dual meanings as metrical balls around, or topological open sets containing the points.
Topological and metric spaces give rise to abstract measures of sets and these lead to Kolmogorov's discovery of probability theory. Here the big idea was to avoid defining randomness, rather go ahead and get relative randomness results from the measures of sets. For example, instead of saying that the relative frequencies tend to a value, the probability, which would be the physical Law of Big Numbers, we can prove that assuming some probabilities, those sequences where the relative frequencies do not tend to these, form only a nil measure, or as it is also said, the tending to the probabilities is "almost" always true. Or in an even better form, the probabilities of not tending to the probabilities is nil, though not impossible, as physically would be. Then different such relative Laws of Big Numbers can be proven with different assumptions. All this was not known when Hilbert posted his famous twenty three problems at the Paris mathematical congress in 1900. Still, it is a mystery why the invariance of openness wasn't listed because its problem was obviously known already. Brouwer only proved it later and called it the Invariance of the Domain, because bounded opens sets were called domains. It is even more illogical because boundedness as we saw is not invariant, but what they meant was the invariance of <u>existing</u> domains around a point. This was just one of the invariance results proved, including the invariance of dimension. The method was very strange too! Homeomorphism was used for the simplest geometrical constructs, that is the distance, triangle, tetrahedra, or simplexes in higher dimensions. This bringing in geometry was actually a trick to use algebraic methods for the coordinates. Later, there were new, more "purely topological" proofs made for the specific invariances, and we'll use one of these too. Still, even in our line of reasoning, the coordinate idea will creep in.
The truth is that we still don't have a crystal clear view, why these invariances are so hard to prove for a normal space. But as we'll see, our normal space has very strange features, so it may not be the "normal" from a higher point of view.

3. Separation and Invariance

D

1.) Two sets are surrounding disjoint, if for any element of any one of them, there is a surrounding where the other has no element.
2.) C is separated if $C = A \cup B$ with A , B surrounding disjoint.
C is unseparated if it is not separated.
3.) C is simply separated if $C = A \cup B$ with A , B surrounding disjoint and both unseparated themselves.
4.) C is a separator if $\overline{C}$ is separated.
C is a simple separator if $\overline{C}$ is simply separated.

R

Two surrounding disjoint sets can approach a common point if it is not element any of them. So in a sense, two such sets "approach each other". However, when a set is the union of two such parts, it still should be regarded as separated, because a homeomorph picture can easily lose this "approach". Of course, with extendedly continuous functions, this could not happen.

T

1.) If A and B are surrounding disjoint and $A \cup B$ is open, then both A , B are open.
2.) If A , B are both unseparated and they are not surrounding disjoint,
then $A \cup B$ is unseparated too.
3.) If A , B are disjoint, each unseparated, and $C = \overline{A \cup B}$,
then $A \cup C$ and $B \cup C$ are not separators.
If C is a simple separator of A and B, then this is also the case.
4.) If C is unseparated and f is a map of it, then f (C) is unseparated too.

P

1.)
Let $P \in A$! There is a surrounding of P fully in $A \cup B$ and so all smaller surroundings are also fully in $A \cup B$. There is a surrounding of P, without B element, so all smaller ones are without B elements too. Thus, here is a surrounding with both features, that is fully in A.
2.)
Suppose $A \cup B$ were separated, that is equal to $C \cup D$ with these surrounding disjoint.
If A and B both had only points from only one of the C , D, then they each were one of these, contradicting that they are not surrounding disjoint. So, one of them, say A, has its points from both C and D. But then, $A = (A \cap C) \cup (A \cap D)$. These two are subsets of C and D and thus, are surrounding disjoint too. This contradicts that A is unseparated.
3.)
Since A , B are disjoint, $A \cup C = \overline{B}$ and $B \cup C = \overline{A}$, so $\overline{A \cup C} = B$ and $\overline{A \cup B} = C$.
4.)
Suppose $f(C) = A \cup B$ with these surrounding disjoint.
Let A* denote the set of all points that are mapped into A and B* the ones into B.
$C = A^* \cup B^*$ so these two are not surrounding disjoint and thus, one of them has a P point approached by the other. But then by continuity f (P) is an element of A or B and approached by the other, contradicting that they are surrounding disjoint.

D

1.) A h [0 , 1] homeomorph picture of [0 , 1] with h (0) = P and h (1) = Q,
is a curve between P and Q.
2.) S is connected if for any $P , Q \in S$, there is a curve between P , Q contained in S.
S is disconnected if it is not connected. S is disconnector if $\overline{S}$ is disconnected.
3.) S is convex if for any $P , Q \in S$, the $\overline{PQ}$ distance is contained in S.

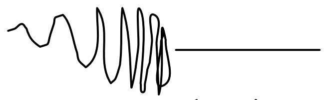
T

1.) [0 , 1] is unseparated.
2.) Connected sets are unseparated too.
3.) Convex sets and complements of bounded convex sets are unseparated.

P

1.) Let [0 , 1] = A ∪ B. At least one of $[0 , \frac{1}{2}]$ or $[\frac{1}{2} , 1]$ again has point of both A and B. Lets choose this and repeat the halving. We get an interval sequence narrowing to a P that is element of A or B and defies them to be surrounding disjoint.
2.) Suppose S were separated but connected! Lets pick a point from the separated members and “draw” a curve between them! Then this curve and [0 , 1] should be separated too.
3.) Convex sets are connected by definition, so unseparated by 2.).
For the complement of a convex set bounded by an I interval, it is enough to prove that any point can be connected to the surface of I. Indeed, then any two can be connected via this surface. We can reach this surface with a single straight line if we connect the P point of the complement with any Q point of the convex set. Indeed, the PQ line can’t cross the convex set, neither in the $\overline{PQ}$ distance, nor on the half line from Q opposite to P. If Q was outside of I, then $\overline{PQ}$ will cross the surface, if Q was inside, then the half line from Q will.

R

An unseparated set doesn’t have to be connected: 

We can easily give though a perfect condition when an unseparated set is connected:

T

1.) If S is unseparated, and every point of it has a surrounding where S is connected, then S is connected too.
2.) Unseparated open sets are connected.
3.) A closed C set is separator, if and only if it is a disconnector.

P

1.) Lets pick a P point of S and regard those points that can be connected to P in S and those that can’t. If S is disconnected, then these two sets are non empty and give S. Since S is unseparated, these two sets can’t be surrounding disjoint, so there will be a point of one of them, that is approached by the other. But, in a surrounding of this point, S should be connected, giving a connection between the two sets and thus, from P to the supposedly unconnectable points too.
2.) Every point of an open set has a surrounding, where the set is full so convex and thus, connected too.
3.) $\overline{C}$ is open and so it is separated, if and only if it is disconnected.

T

Jordan

1.) Compact Separator Invariance
If C is a compact separator and h a homeomorphism of it, then h (C) is separator too.
2.) Sphere Separation Invariance
If S is a sphere and h a homeomorphism of it, then h (S) is a separator.
3.) Simple Compact Separator Invariance
In 1.), if C is simple separator, then h (C) is simple too.
4.) Sphere Simple Separation Invariance
In 2.) h (S) is simple separator.

R

2.) is special case of 1.) and 4.) is of 3.) because the sphere's inside ball is convex, so both this and the outside are unseparated. 3.) and even 4.) are very hard to prove.
1.) could be called as Compact Disconnector Invariance, by 3.) of the theorem before Jordan. Actually this meaning will be proved along this book.
If C is not compact, only closed or bounded, then h (C) doesn't have to be a separator.
For example, an infinite unbounded plane, is closed and separates the space, but it is homeomorph with a disc without its perimeter. This is not closed but bounded and doesn't separate the space. To get an exact hom. for this example, we can simply regard a P point's d distance from an O origin and h (P) should be on the same line, but at $\frac{1}{1+\frac{1}{d}}$ distance.

To prove the Invariance of Openness, we need simple separation by a sphere's picture, that is 4.). But since we only want to rely on 1.) and 2.), we need a trick to get simpleness from some additional condition, that is true at the Invariance of Openness. This condition is that the h homeomorphism is defined not only for the S sphere, but for its inside B ball too. Now, if a homeomorphism is defined for not only a sphere or simple separator, but also for the inside of these, then h maps the inside into the inside. We however, won't go into the distinction, of the inside and the outside right now, only in the next section. So, we'll merely claim that if h is defined for one of the simply separated members, then the picture will be also one of the simply separated members. This theorem was first stated by Bolzano in analysis, saying that a continuous function of real numbers picks up all the numbers in between two values. The picture of the two points is again two points, so the inside is trivial. For higher dimensions however, the inside or outside relies on separation, so usually the theorem is only stated in a weaker version for homeomorphisms:

T

Bolzano

1.) Full range for compact simple separator.
If C is a compact simple separator with $\overline{C} = A \cup B$, where A , B are surrounding disjoint and unseparated, and h is a homeomorphism of $B \cup C$,
then h (C) is a simple separator too, namely $\overline{h(C)} = h(B) \cup \overline{h(B \cup C)}$ with these being surrounding disjoint and unseparated.

2.) Ball's full range.
If S is a sphere, B the open ball inside of it, and h a homeomorphism of $B \cup S$,
then $\overline{h(S)} = h(B) \cup \overline{h(B \cup S)}$ with these being surrounding disjoint and unseparated.

P

2.) is special case of 1.) So, it's enough to prove 1.):
h (B) is unseparated because it is a continuous picture of the unseparated B. (first T 4.)).
h (C) is a separator by Jordan 1.), so $\overline{h(C)}$ is separated.
$B \cup C$ is not separator because C was simple separator. (first T 3.)). So,
$h(B \cup C)$ is not separator either by Jordan 1.) reverse. So, $\overline{h(B \cup C)}$ is unseparated.
$h(C) = h(B \cup C) - h(B) = h(B \cup C) \cap \overline{h(B)}$, so $\overline{h(C)} = h(B) \cup \overline{h(B \cup C)}$.
The left side is separated. The two members on the right are unseparated, so they must be surrounding disjoint, otherwise the left side were unseparated. (first T 2.))

T

Brouwer:

1.) Inner Point Invariance
If h is a homeomorphism and P an inner point of its domain,
then f (P) is inner point of the range too.

2.) Openness Invariance
If h is a homeomorphism of an open A set, then h (A) is open too.

P 2.) is obvious by 1.) so enough to prove 1.):
There is an S sphere and a B ball inside of it, so that $P \in B$ and $B \cup S \subseteq$ domain.
By Ball's full range: $\overline{h(S)} = h(B) \cup \overline{h(B \cup S)}$ with these being surrounding disjoint.
S is compact and so h (S) is too. Then, $\overline{h(S)}$ is open and so h (B) is open too. (first T 1.))
Thus, the $h(P) \in h(B)$ point has a surrounding fully in h (B), which is in the range.

4. Components

R In this section, we still don't start the long proof of the Jordan theorem, rather elaborate on the used Bolzano theorem. Firstly, we didn't go into the distinction of inner and outer members at a simple separator. Now we show how this can be done, in fact we define the components for non simply separated sets too. Then, we show two alternative ideas to get the simplicity. One, replacing Bolzano, the other, extending it by avoiding its condition of h being defined inside. None of these ideas will work, but their failures will be shown by exciting counter examples.

D

1.) For any $P \in S$, let S_P denote the union of all unseparated subsets of S, that have P as element. By first T 2.) of previous section, this is unseparated too.
2.) Lets call a subset of S a component if it is unseparated and there is no wider unseparated subset of S. Thus, S_P is the component of S, containing P. This also implies that:
Every set is the union of surrounding disjoint components.

To be continued . . .